RICCATI EQUATIONS FOR NONSYMMETRIC AND NONDISSIPATIVE HYPERBOLIC SYSTEMS WITH L^2-BOUNDARY CONTROLS

BY

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A quadratic cost boundary control problem for a hyperbolic system on a domain $\Omega \subset R^m$ with a smooth boundary Γ is discussed.

Let F and G be given self-adjoint, positive-definite bounded operators on $L^2(\Omega)$.

Our problem is to minimize the functional cost $J(u,y) = |u|^2_{[0,T]x\Gamma} + (y,Fy)_{[0,T]x\Omega} + (y(T),Gy(T))_\Omega$ over controls $u \in L^2([0,T]x\Gamma)$ where y is the solution to the nonsymmetric and nondissipative hyperbolic system of first order:

$$\frac{\partial y}{\partial t} = A(x,\partial)y \quad \text{in } [0,T]x\Omega$$

$$My = u \quad \text{in } [0,T]x\Gamma$$

$$y(0) = y_o \quad \text{in } \Omega$$

with $y_o \in L^2(\Omega)$.

With some technical assumptions on the differential operator $A(x,\partial)$ and the boundary operator $M(x)$, we obtain a semigroup representation for the hyperbolic system.

It is then shown that our problem admits a unique optimal control u^0 in $L^2([0,T]\times\Gamma)$ with the corresponding trajectory y_0 in $C([0,T]; L^2(\Omega))$. We show that the optimal control u^0 can be expressed in a feedback form via a Riccati operator on $L^2(\Omega)$; i.e.,

$$u^0(t) = BP(t)y^0(t) \qquad \text{a.e. } t \in [0,T]$$

where B is an unbounded operator from $L^2(\Omega)$ into $L^2(\Gamma)$ and $P(t)$ is a Riccati operator on $L^2(\Omega)$ which satisfies an appropriate Riccati equation.

Under some smoothness assumptions on F and G, we give additional regularity results for the optimal solution and show that the Riccati operator $P(t)$ is the unique solution to the appropriate Riccati equations within a certain class of one paramater families of operators on $L^2(\Omega)$.

Finally, it is shown that the Riccati operator $P(t)$ for the L^2-problem in which F and G are assumed to have no smoothing properties can be determined as the strong limit, uniformly in t, of a sequence of solutions to the Riccati equations corresponding to regularized problems in which F and G are assumed to have some smoothing properties.

CHAPTER I
INTRODUCTION

1.1 Notations

Let Ω be an open domain in R^m with smooth boundary. We denote by $C_o^\infty(\Omega;R^k)$ the space of infinitely differentiable maps from Ω into R^k with compact supports, and by $C_{(o)}^\infty(\Omega,R^k)$ the restriction of $C_o^\infty(R^m;R^k)$ to $\overline{\Omega}$. Let $L^2(\Omega;R^k)$ be the space of Lebesque square integrable maps from Ω into R^k with natural inner product $(\cdot,\cdot)_{L^2(\Omega;R^k)}$ and norm $|\cdot|_{L^2(\Omega;R^k)}$. For convenience of notation, we write simply $L^2(\Omega)$, $(\cdot,\cdot)_\Omega$ and $|\cdot|_\Omega$ if there is no confusion.

Some Sobolev spaces are defined as follows: For $s > 0$ integer, let $H^s(\Omega;R^k)$ be the completion of $C_{(o)}^\infty(\Omega,R^k)$ with respect to the norm

$$(1.1)\qquad |y|_{s,\Omega}^2 = \sum_{|\alpha| < R_s} |\partial^\alpha y|_\Omega^2 = \sum_{|\alpha| < R_s} \int_\Omega |\partial^\alpha y|^2 dx$$

where $\alpha = (\alpha_1,\ldots,\alpha_m)$ multi-indices, $|\alpha| = \alpha_1 + \ldots + \alpha_m$, and $\partial^\alpha = \dfrac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \dfrac{\partial^{\alpha_m}}{\partial x_m^{\alpha_m}}$.
Let $H_o^s(\Omega;R^k)$ be the completion of $C_o^\infty(\Omega;R^k)$ with respect to (1.1). Similarly Sobolev spaces of noninteger orders are

defined by interpolation as in [13]. As a matter of convenience, $H^s(\Omega;R^k)$ and $H_o^s(\Omega;R^k)$ are abbreviated to $H^s(\Omega)$ and $H_o^s(\Omega)$, repsectively, if no confusion arises.

Let J be an interval and X be a Hilbert space. We introduce the following spaces:

$C(J;X)$ is the space of continuous maps from J into X, with supremum norm,

$L^\infty(J;X)$ is the space of measurable and essentially bounded maps from J into X, with essential supremum norm,

$L^1(J;X)$ is the space of Lebesque integrable maps from J into X, with norm $|y|_{L^1(J;X)} = \int_J |y(t)|_X dt$,

$L^2(J;X)$ is the space of Lebesque square integrable maps from J into X, with norm $|y|^2_{L^2(J;X)} = \int_J |y(t)|^2_X dt$, and

$H^1(J;X) = \{y | y \in L^2(J;X) \text{ and } \frac{d}{dt} y \in L^2(J;X)\}$ with norm

$$|y|^2_{H^1(J;X)} = |y|^2_{L^2(J;X)} + |\frac{d}{dt} y|^2_{L^2(J;X)}.$$

In view of Fubini's theorem, we have $L^2(J;L^2(\Omega)) = L^2(J\times\Omega)$. For a bounded linear operator P from a Banach space X into a Banach space Y, we denote by $\|P\|_{X\to Y} = \sup_{|x|_X=1} |Px|_Y$ the operator norm. If there is no confusion, we write simply $\|P\|$. We also denote by X' the dual space of a Banach space X.

1.2 Problem Formualtion and Preliminary Material

Let Ω be an open bounded domain in R^m with smooth boundary Γ.

We consider a first order differential operator of the form

$$(1.2) \quad A(x,\partial) = \sum_{j=1}^{m} A_j(x)\partial_j + C(x),$$

where $\partial_j = \partial/\partial x_j$ and the coefficients $A_j(x)$ and $C(x)$ are smooth n x n matrix-valued functions defined on Ω.

We make the following assumptions on the operator $A(x,\partial)$:

(A.1) (Strict Hyperbolicity)

For all nonzero vectors $(\omega_1,\ldots\omega_m)$ in R^m and all x in $\overline{\Omega}$, the matrices $\sum_{j=1}^{m} A_j(x)\omega_j$ have n distinct real eigenvalues.

For x in Γ, let $A^N(x) = \sum_{j=1}^{m} A_j(x)n_j(x)$ where $(n_1(x),\ldots,n_m(x))$ is the inward unit mormal to the boundary Γ at x.

(A.2) (Noncharacteristic Boundary)

The matrix $A^N(X)$ is nonsingular for each x in Γ.

Remark. From the continuity of the A_j's and assumptions (A.1), (A.2), we deduce that $A^N(x)$ has n distinct real eigenvalues different from zero for each x in Γ. Henceforth, we assume, without loss of generality, that $A^N(x)$ has k distinct negative eigenvalues and (n-k) distinct positive eigenvalues for each x.

We impose a boundary condition on the operator $A(x,\partial)$ with the aid of a boundary operator $M(x)$ which is a smooth $k \times n$ matrix-valued function defined on Γ of rank k and annihilates all vectors in the null space of $A^N(x)$ for each $x \in \Gamma$.

We now consider the following initial boundary value problem: For $T > 0$, $f \in L^2([0,T] \times \Omega)$, $u \in L^2([0,T] \times \Gamma)$ and $y_o \in L^2(\Omega)$,

$$(1.3) \qquad \begin{aligned} &\partial_t y = A(x,\partial)y + f && \text{in } [0,T] \times \Omega \\ &My = u && \text{in } [0,T] \times \Gamma \\ &y(0) = y_o && \text{in } \Omega \end{aligned}$$

Let us denote $Q = [0,T] \times \Omega$ and $\Sigma = [0,T] \times \Gamma$.

We recall the definition of strong solutions to (1.3).

Definition. A function y in $L^2(Q)$ is said to be a strong solution to (1.3) if there exists a sequence $\{y_n\}_n$ in $C^\infty_{(o)}(Q)$ and v in $L^2(\Sigma)$ such that $Mv = u$,

$$|y_n - y|_Q \to 0$$

$$|y_n - v|_\Sigma \to 0$$

$$|\partial_t y_n - A(x,\partial)y_n - f|_Q \to 0$$

and

$$|y_n(0) - y_o|_\Omega \to 0 \text{ as } n \to \infty.$$

It is well known [15] that by using a partition of unity and change of local coordinates, the above mixed problem (1.3) can be transformed into a problem on a half-space.

Thus we assume, without loss of generality, that Ω is the half-space $\{x|x = (x_1,x_2,\dots,x_m) \in R^m,\ x_1 > 0\}$ with $\Gamma = \{x|x \in R^m,\ x_1 = 0\}$. For the half-space Ω, the matrix $A^N(x)$ in (A.2) becomes $A_1(x)$ for each x in Γ. In view of finite speed of propagation in hyperbolic systems, we require, without loss of generality, that the matrices A_j's, C and M are constant outside a compact set in Ω [15].

From the assumptions (A.1) and (A.2), by a smooth change of coordinates, A_1 can be written in the following standard form:

$$(1.4) \quad A_1(x) = \begin{bmatrix} A_1^-(x) & 0 \\ 0 & A_1^+(x) \end{bmatrix} \quad \text{for } x \in \overline{\Omega} \text{ where}$$

$$A_1^-(x) = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_k \end{bmatrix} \quad \text{with negative funtions } a_1,\dots,a_k,$$

and $A_1^+(x) = \begin{bmatrix} a_{k+1} & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$ with positive functions $a_{k+1},\dots,a_n$.

For simplicity, we use the following notation:

$x = (x_1 \vdots x_2,\dots,x_m) = (x_1,x^1)$ and

$y = (y_1,\dots,y_k \vdots y_{k+1},\dots,y_n) = (y^-,\ y^+)$.

As to the boundary operator M for the half-space Ω, we simplify as follows. Let us partition M as $[H \vdots N]$ where H is a k x k matrix and N is a k x (n-k) matrix.

With $A_1(x)$ in the standard form (1.4), it is well known that a necessary condition for (1.3) to be well-posed is that H be nonsingular [5, 15]. Since M and $H^{-1}M$ impose essentially the same boundary conditions, we assume that

(1.5) $M = [I \vdots N]$ where I is the k x k identity matrix.

In order to make the mixed problem (1.3) well-posed in the Kreiss' sense, we need to introduce the so called "Kreiss' condition" formulated in terms of eigenvalues and generalized eigenvalues as follows.

For a moment, we assume that all the coefficient matrices A_j's and M in (1.3) are constant, and we then consider the boundary value problem:

$$(1.6) \qquad \partial_t y = \sum_{j=1}^{m} A_j \partial_j y \quad \text{in } Q$$

$$My = 0 \qquad \text{in } \Sigma$$

We take the Fourier and Laplace transforms of (1.6) with respect to the tangential variables $x^1 = (x_2, \ldots, x_m)$ and the time variable t with the dual variables $\omega = (\omega_2, \ldots, \omega_m)$ and $s = r + iq$, respectively. We then obtain

$$(1.7) \qquad \frac{d\hat{y}}{dx_1} = A_1^{-1} (sI - i \sum_{j=2}^{m} A_j \omega_j)\hat{y} \qquad \text{for } x_1 > 0$$

$$(1.8) \qquad M\hat{y} = 0 \qquad \text{for } x_1 = 0$$

where $\hat{y}$ is the Fourier and Laplace transforms of y.

We recall the following definition from [8].

Definition. A complex number $s = r + iq$ is said to be an eigenvalue of (1.3) if there exists a nontrivial solution $\phi(x_1)$ of the ordinary differential equation (1.7) and (1.8) in $L^2([0,\infty);R^n)$.

Since $A_1^{-1}(sI - i \sum_{j=2}^{m} A_j\omega_j)$ has exactly k eigenvalues with negative real parts for Re $s > 0$ [8], it is well known that any solution ϕ of (1.7) in $L^2([0,\infty))$ can be expressed as a linear combination of k linearly independent normalized solutions $\phi_1(s;\cdot),\ldots,\phi_k(s;\cdot)$ in $L^2([0,\infty))$. Hence, from (1.8), a number s with positive real part is an eigenvalue of (1.3) if $\sum_{j=1}^{k} M\,\phi_j(s;\cdot)C_j = 0$ for some nonzero vector $(c_1,\ldots,c_k)$; i.e., the square matrix $(M\phi_1,(s;\cdot),\ldots,M\phi_k(s;\cdot))$ is singular.

Definition. A pure imaginary number iq is said to be a generalized eigenvalue of (1.3) if the matrix $(M\phi_1(s;\cdot),\ldots,M\phi_k(s;\cdot))$ approaches a singular matrix as $r \to 0^+$, where $s = r + iq$.

We assume the following condition.

(A.3) (Kreiss' condition)
For each boundary point $(0,x^1)$ in Γ, the constant coefficient problem (1.3) that arises by freezing the coefficients A_j's and M at $(0,x^1)$ has no eigenvalues or generalized eigenvalues with nonnegative real parts.

We refer to [8, 15, 18] for other equivalent conditions.

Now we are in a position to recall basic results on well-posedness and regularity for (1.3) from [15].

Theorem 1.1 Well-posedness.

Assume (A.1) ~ (A.3).

Then, for $f \in L^2(Q)$, $y_o \in L^2(\Omega)$ and $u \in L^2(\Sigma)$, the variable coefficient mixed problem (1.3) admits a unique strong solution y in $L^2(Q)$. Furthermore, the following estimate holds:

$$(1.9) \qquad |y(t)|_\Omega + |y|_Q + |y|_\Sigma$$

$$\le C_T \, [|f|_Q + |u|_\Sigma + |y_o|_\Omega] \text{ for } 0 \le t \le T,$$

where C_T is a constant independent on f, y_o, u and t.

Theorem 1.2 Regularity.

Assume (A.1) ~ (A.3).

Let s be any positive integer. Suppose that $f \in H^s(Q)$, $y_o \in H^s_o(\Omega)$ and $u \in H^s(\Sigma)$ satisfy the following compatibility conditions:

$$\partial_t^j f|_{t=0} = 0 \text{ and } \partial_t^j u|_{t=0} = 0 \text{ for all integers } 0 \le j < s.$$

Then the solution y to (1.3) has the following regularity:

$$y \in H^s(Q),\ y(t) \in H^s(\Omega) \text{ and } y|_\Gamma \in H^s(\Sigma),\ 0 \le t \le T.$$

Moreover for $0 \le t \le T$,

$$(1.10)\qquad |y(t)|_{s,\Omega} + |y|_{s,Q} + |y|_{s,\Sigma}$$
$$\leq C_s\,[|f|_{s,Q} + |y_o|_{s,\Omega} + |u|_{s,\Sigma}]$$

where C_s is a constant independent on f, y_o, u and t.

With theorem 1.1 and 1.2 at hand, we are now in a position to formulate our optimal control problem.

We introduce the state equation:

$$(1.11)\qquad \begin{aligned} &\partial_t y = A(x,\partial)y && \text{in } Q \\ &My = u && \text{in } \Sigma \\ &y(0) = y_o && \text{in } \Omega \end{aligned}$$

where $A(x,\partial)$ and M are assumed as in theorem 1.1, $y_o \in L^2(\Omega)$ and $u \in L^2(\Sigma)$.

From theorem 1.1, for any function (control) u in $L^2(\Sigma)$, there corresponds a unique solution (trajectory) to (1.11) which will be denoted by y^u.

Let F and G be any bounded, self-adjoint and positive-definite operators on $L^2(\Omega)$.

We consider the following functional cost:

$$(1.12)\qquad J(u,y^u) = \frac{1}{2}\,[|u|^2_\Sigma + (y^u,Fy^u)_Q + (y^u(T),Gy^u(T))_\Omega]$$

$$\text{for } u \text{ in } L^2(\Sigma).$$

We note, from theorem 1.1, that (1.12) is well-defined.

Remark. A more general functional cost such as $J(u,y^u) = \frac{1}{2}\,[|u|_\Sigma^2 + (y^u - y^d, F(y^u - y^d))_Q + (y^u(T) - y^d(T), G(y^u(T) - y^d(T)))_\Omega]$, where y^d is a given function in $L^2(Q)$ with $y^d(T) \in L^2(\Omega)$, can be treated in the study which follows. However, for the sake of simplicity of notation, we shall restrict ourselves in what follows to functional cost (1.12).

Optimal Control Problem:

(C.P.) Minimize the functional cost $J(u,y^u)$ over u in $L^2(\Sigma)$ subject to the state equation (1.11).

Our basic questions are

(i) Does the (C.P.) admit a unique optimal control in $L^2(\Sigma)$?

(ii) If yes in (i), can the optimal control be expressed in a Riccati feedback form?

Our aim is to give affirmative answers to the above questions. To elaborate (ii), our goal in this work is to express the optimal control u^o in a feedback form;

$$u^o(t) = BP(t)y^o(t), \qquad \text{a.e. } t \in [0,T]$$

where B is an unbounded operator from $L^2(\Omega)$ into $L^2(\Gamma)$, and $P(t)$ is a Riccati operator on $L^2(\Omega)$ which satisfies a corresponding Riccati equation.

More details are presented in sections 1.3 and 1.4.

1.3 Survey of Literature

A number of authors have studied optimal control problems for dynamics which include first order hyperbolic systems. Here we briefly mention some of the results already obtained in this field and compare them with the results obtained in this thesis.

In the fundamental treatise by Lions [12] on optimal control problems for partial differential equations, boundary control problems for second order hyperbolic equations (not for first order systems) are briefly mentioned under the assumption of smooth controls.

Russell [16] has studied boundary control problems for first order symmetric hyperbolic systems in one spatial variable, through the method of characteristics. It is well known that the method of characteristics cannot be applied to problems in several spatial variables.

Curtain and Pritchard [2, 3] have studied optimal control problems described by evolution equations. When they specialize their results to boundary control problems for first order hyperbolic systems, it is needed to assume that the control spaces are $L^2([0,T]; H^{\frac{1}{2}}(\Gamma))$ (rather than $L^2([0,T]; L_2(\Gamma)))$.

The work that is closest in spirit to ours is a paper by Vinter and Johnson [18], in which they discuss Riccati equations for hyperbolic systems with boundary controls.

A brief summary of the results in [18] is provided here. Vinter and Johnson [18] consider functional cost (1.12) under the assumption F = 0, that is,

$$(1.13) \qquad J(u,y^u) = |u|^2_\Sigma + (y^u(T), Gy^u(T))_\Omega .$$

By using a variational techniques as in [12], they characterize the optimal control in terms of the solution z of the two-point boundary value problem;

$$(1.14) \qquad \begin{aligned} \partial_t z &= -A^* z - Fy && \text{in } Q \\ M^* z &= 0 && \text{in } \Sigma \\ z(T) &= Gy(T) && \text{in } \Omega \end{aligned}$$

and

$$\begin{aligned} \partial_t y &= A(x,\partial)y && \text{in } Q \\ My &= u && \text{in } \Sigma \\ y(0) &= y_0 && \text{in } \Omega \end{aligned}$$

where A^* is the formal adjoint of the operator $A(x,\partial)$ and M^* is the adjoint boundary operator of M.

Only under the supplemental assumption that G maps $L^2(\Omega)$ into $H^1_0(\Omega)$, they derive an appropriate Riccati differential equation, and then synthesize the optimal control in a feedback form. The main technical tools in [18], besides variational techniques, are the regularity theorems from [15], together with trace theorems.

To summarize, the following comments pertaining to [18] are in order:

(i) Operator G in (1.13) is assumed to carry $L^2(\Omega)$ into $H_o^1(\Omega)$. This means that the penalization of the terminal state y(T) occurs in a space of distributions, instead of L^2-space.

(ii) Functional cost (1.13) does not penalize the trajectory, i.e. does not contain the term $(y,Fy)_Q$ of (1.12).

However, the penalization of the trajectory is important, because it is precisely responsible for the minimization of the whole energy of the system, which in turn is a primal task of engineering implementations. The techniques of [18] cannot be extended to treat the important case when the trajectory or the terminal state is penalized in the physically meaningful L^2-spaces. This is due to the fact that the regularity theorems in [15] are not applicable to the case of $F \neq 0$. To understand this point better, let us mention the argument of [18] which relies on differentiability of the solution z of (1.14).

In order to deduce differentiabilty of the solution z by applying Rauch's regularity theorems, some compatibility conditions are needed for the terms Fy and Gy(T) in (1.14). But without assuming $F = 0$, the compatibilty conditions cannot be satisfied. For this reason, one should use different techniques from [18], in order to overcome the limitations discussed in (i) and (ii).

Motivated by the physical significance of the problem, we study Riccati feedback synthesis with boundary controls in L^2-spaces; i.e., the trajectory, the terminal state and the control are all penalized in L^2-topologies. This is a distinguishing feature of our work from the existing literature. In this sense, we extend and complement [18].

More precisely, we do obtain feedback realization of the optimal control as postulated in (C.P.), without any smoothness assumptions on the operators F and G. It is also shown that the feedback operator P(t) does satisfy a Riccati integral equation. Our techniques different from those in [18] are based on more refined analysis of regularity of the solution to the mixed problem (1.3). This analysis leads to a more accurate description of the trace of the Riccati operator P(t), which is a necessary ingredient in the derivation of Riccati equations.

In this analysis, we use a combination of partial differential equation- and semigroup- methods together with so-called "direct approach" as in [9,10]. It should be pointed out that penalizations in L^2-spaces, instead of distributions, lead to major mathematical difficulties related to the trace of the Riccati operator P(t).

The Riccati differential equation for the original L^2-problem may not have meaning, even in the distributional sense. Here we use the regularization technique to overcome the difficulties.

In a regularized problem, we show that the Riccati operator P(t) is the unique solution to appropriate Riccati equations in a certain class of operators on $L^2(\Omega)$. The original L^2-problem is then recovered as a limit of the regularized problems. This way, we show that the Riccati operator for the original problem is determined as a strong limit of Riccati operators for a sequence of regularized problems.

From the numerical point of view, the regularization results can be considered as a tool to construct a convergent algorithm for computing the Riccati operator of the original L^2-problem. Finally, the question of regularity for the optimal solutions to our control problem is answered under some assumption on F and G. Precise statements of our results are given in the next section.

1.4 Main Results

In order to state our main results, we need some preliminary material.

Let A be the operator defined by

(1.15) $Ay = A(x,\partial)y$ for y in the domain $D(A)$

where $D(A) = \{y \in L^2(\Omega) \mid Ay \text{ in } L^2(\Omega) \text{ and } My|_\Gamma = 0\}$. It is well known [15, 18] that A generates a strongly continuous semigroup $S(t)$ on $L^2(\Omega)$.

We next introduce the "Dirichlet" map "D" (a natural extension from the boundary Γ into the interior Ω) defined by "Du = y" where

$$(1.16) \qquad \begin{aligned} A(x,\partial)y &= ky \quad \text{in } \Omega \\ My|_\Gamma &= u \quad \text{in } \Gamma \end{aligned}$$

for some sufficiently large constant $k > 0$ (Dependence of D on k will not be explicitly indicated). We show in section 2.1 that the above Dirichlet map D is a bounded linear operator from $L^2(\Gamma)$ into $L^2(\Omega)$ (theorem 2.1).

With the help of the operators A and D, we give an abstract semigroup model for the solution of (1.9). In fact, we define an operator L as

$$(1.17) \qquad (Lu)(t) = (A-kI)\int_0^t S(t-s)Du(s)ds \text{ for } 0 \leq t \leq T$$

and u in the domain of L which is a dense subspace of $L^2(\Sigma)$. Then we prove, in section 2.2, the following theorem.

Theorem 2.2 Semigroup representation.

(i) The operator L is a linear bounded operator from $L^2(\Sigma)$ into $C([0,T]; L^2(\Omega))$.

(ii) The solution y to the mixed problem (1.9) is given by the semigroup representation:

$$y(t) = S(t)y_0 - (Lu)(t) \text{ for } 0 \leq t \leq T.$$

In section 3.1, we show that our control problem (C.P.) admits a unique optimal control u^0 in $L^2(\Sigma)$.

Now we are in a position to state our main results (Theorem 3.2, 4.1, 5.1, 5.2, and 5.3).

Theorem 3.2 Regularity of optimal solution.

Under the assumption that F and G are compact operators on $H^s(\Omega)$ for all $0 \leq s < \frac{1}{2}$, it is shown that

(i) u^o is in $H^s(\Sigma)$

(ii) y^o is in $H^s(Q)$

for any initial data y_o in $H^s(\Omega)$.

Let us denote by $\Phi(t,s)$, $0 \leq s \leq t \leq T$, the evolution operator on $L^2(\Omega)$ which describes the evolution of the optimal trajectory y^o.

Theorem 4.1 Feedback synthesis of optimal control.

The optimal control u^o can be expressed as follows.

(i) $u^o(t) = BP(t)y^o(t)$ for t in $[0,T]$,

where $B = D^*(A^*-kI)$ which is given by $By = A_1^- y^- |_\Gamma$ for y in $D(A^*)$.

(ii) P(t) is a bounded operator on $L^2(\Omega)$, satisfying the following Riccati integral equation (the first Riccati equation):

For x,y in $L^2(\Omega)$, t in $[0,T]$,

$$\text{(I R.E.)} \quad (P(t)x,y)_\Omega = \int_t^T (\Phi(s,t)x, F\Phi(s,t)y)_\Omega ds$$

$$+ \int_t^T (BP(s)\Phi(s,t)x, BP(s)\Phi(s,t)y)_\Gamma ds$$

$$+ (\Phi(T,t)x, G\Phi(T,t)y)_\Omega .$$

As a corollary of (I R.E.), we obtain that P(t) is self-adjoint and positive-definite.

Control problems with smooth observations are treated in theorems 5.1 and 5.2.

Theorem 5.1 Riccati equations.

Under the assumption that F and G map $L^2(\Omega)$ into the domain $D(A^*)$, $P(t)$ satisfies the following Riccati equations.

(i) (Riccati integral equation, which we call the second Riccati equation):

For x,y in $L^2(\Omega)$, $0 \leq t \leq T$,

$$\text{(II R.E.)} \quad (P(t)x,y)_\Omega = \int_t^T (S(s-t)x,FS(s-t)y)_\Omega ds - \int_t^T (BP(s)S(s-t)x,BP(s)S(s-t)y)_\Gamma ds + (S(T-t)x,GS(T-t)y)_\Omega .$$

(ii) (Riccati differential equation):

For x,y in $D(A)$ and a.e. t in $[0,T]$,

$$\text{(D.R.E.)} \quad \frac{d}{dt}(P(t)x,y)_\Omega = -(x,Fy)_\Omega - (P(t)Ax,y)_\Omega - (P(t)x,Ay)_\Omega + (BP(t)x,BP(t)y)_\Gamma$$

with terminal condition $P(T) = G$.

Theorem 5.2 Uniqueness.

Under the same assumptions as in theorem 5.1, $P(t)$ is the unique solution to the Riccati equations in theorem 5.1 within the class $\mathfrak{P}$, where $\mathfrak{P}$ is the class of one parameter families $\{\overline{P}(t)\}_{t\geq 0}$ of self-adjoint, positive-definite and bounded operators on $L^2(\Omega)$ such that

$B\overline{P}(\cdot)$ is bounded from $L^2(\Omega)$ into $L^\infty([0,T];\ L^2(\Gamma))$

and $B\overline{P}(\cdot)S(\cdot)B^*$ is bounded from $L^2(\Gamma)$ into $L^2(\Sigma)$.

Let P_n be the corresponding Riccati operators for the control problems with F_n and G_n in the functional cost $J(u,y)$.

Finally, we obtain the following convergence.

Theorem 5.3 Convergence.

Let F and G be any self-adjoint, positive-definite and bounded operators on $L^2(\Omega)$.

Suppose that F_n and G_n map $L^2(\Omega)$ into $D(A^*)$, and that $F_n \to F$ and $G_n \to G$ strongly in $L^2(\Omega)$ as $n \to \infty$.

Then, $P_n(t) \to P(t)$ strongly in $L^2(\Omega)$ as $n \to \infty$ uniformly in t.

Moreover, for x,y in $L^2(\Omega)$ and uniformly in t,

$$(P(t)x,y)_\Omega = \int_t^T (S(s-t)x,FS(s-t)y)_\Omega ds + (S(T-t)x,GS(T-t)y)_\Omega$$

$$- \lim_{n\to\infty} \int_t^T (BP_n(s)S(s-t)x,BP_n(s)S(s-t)y)_\Gamma ds.$$

Remark. The significance of theorem 5.3 is two fold:

(i) it serves as a tool for proving theorem 4.1 (ii),

(ii) it enables to prove convergence of an effective numerical algorithm to determine the feedback operator P(t) for the L^2-problem.

The rest of the paper is devoted to proofs of the main results as follows.

In section 2.1, we prove the existence and regularity of the "Dircihlet map" D (theorem 2.1). Section 2.2 gives the proof of theorem 2.2. In section 3.1, we explicitly characterize the optimal control u^o and optimal trajectory y^o (theorem 3.1). In section 3.2, we give regularity results for the optimal solutions (theorem 3.2 and theorem 3.3). In section 4.1, the evolution operator "Φ" is introduced (lemma 4.2). Section 4.2 gives the proof of theorem 4.1 (i) and a partial proof of second part (ii). In section 5.1 and 5.2, we give the proof of theorem 5.1 and 5.2. Finally, in section 5.3, we give the proof of theorem 5.3 and we complete the proof of theorem 4.1 (ii). Also, examples of sequences $\{F_n\}$ and $\{G_n\}$ in theorem 5.3 are given.

CHAPTER II
SEMIGROUP REPRESENTATION FOR HYPERBOLIC SYSTEMS

2.1 Dirichlet Map

In order to introduce a semigroup representation for the solution to (1.11), we need to extend functions defined on Γ to functions on Ω.

An operator which maps boundary functions into specially defined interior extensions is said to be a "Dirichlet map."

In this section, we show that the Dirichlet map "D" defined in (1.16) is a bounded linear operator from $L^2(\Gamma)$ into $L^2(\Omega)$.

We consider the following boundary value problem:

$$(1.16) \qquad \begin{aligned} A(x,\partial)y &= ky \quad \text{in } \Omega \\ My &= u \quad \text{in } \Gamma \end{aligned}$$

for u in $L^2(\Gamma)$ where k is constant > 0. The existence of our Dirichlet map D follows from the next theorem.

Theorem 2.1.

There exists a constant $k > 0$ large enough, such that for any $u \in L^2(\Sigma)$, (1.16) has a unique solution y in $L^2(\Omega)$.

Furthermore, the following energy inequality holds:

$$(2.1) \qquad |y|_\Omega + |y|_\Gamma \leqslant c|u|_\Gamma$$

for a constant c independent on u.
The proof will be given later.

Corollary. The Dirichlet map D is a bounded linear operator from $L^2(\Gamma)$ into $L^2(\Omega)$.

Remark. The above result was first proven in [11] for symmetric problems with dissipative boundary conditions, and then was extended to symmetrizable problems with dissipative boundary conditions in [4]. In our nonsymmetric and nondissipative case, the proof of theorem 2.1 will be given by using "Kreiss' symmmetrizer technique" as in [8].

Proof of Theorem 2.1. We may assume, without loss of generality, that $C(x) = 0$ identically. We shall first prove the energy inequality (2.1) and then the existence of the solution to (1.16). Suppose for a moment that the coefficient matrices A_j's and M are constant frozen at a boundary point $(0,x^1)$ in Γ. We apply the Fourier transform to (1.16) for the tangential variables x^1 with the dual variables $\omega = (\omega_2,\ldots,\omega_m)$. Then (1.16) becomes

$$(2.2) \qquad A_1 \frac{d\hat{y}}{dx_1} = [kI-i \sum_{j=2}^{m} A_j\omega_j]\hat{y} \qquad \text{for } x_1 > 0$$

$$M\hat{y} = \hat{u} \qquad \text{for } x_1 = 0.$$

Since A_1 is nonsingular, (2.2) becomes

$$\frac{d\hat{y}}{dx_1} = A_1^{-1}\,[kI - i\sum_{j=2}^{m} A_j\omega_j]\hat{y} \qquad \text{for } x_1 > 0$$

$$M\hat{y} = \hat{u} \qquad \text{for } x_1 = 0$$

which is the same sort of problem Kreiss studied in [8].

Now we may rewrite (2.2) in pseudo-differential form as

$$(2.3) \qquad A_1(x)\,\frac{d\hat{y}}{dx_1} = B(x,\omega)\hat{y}\ , \quad x_1 > 0$$

$$M(x^1)\hat{y} = \hat{u}, \qquad x_1 = 0$$

where $B(x,\omega)$ is a symbol of the pseudo-differential operator (of first order) corresponding to the differential operator $kI - \sum_{j=2}^{m} A_j(x)\partial_j$.

Since A_1 is nonsingular, we have

$$(2.4) \qquad \frac{d\hat{y}}{dx_1} = V(x,\omega)\hat{y}\ , \quad x_1 > 0$$

$$\hat{M}\hat{y} = u, \qquad x_1 = 0$$

where $V(x,\omega) = A_1^{-1}(x)B(x,\omega)$.

Now we use the Kreiss symmetrizer which is constructed locally in conical neighborhoods of $(x,\omega,0)$ [8].

We recall the following results from [8].

Lemma 2.1. There exists a symmetrizer for (2.4) which is a pseudo-differential operator (of order zero) whose symbol $\hat{R}(x,\omega,k)$ with L^2-norm independent of k satisfies

(i) $\hat{R}$ is Hermitian,

(ii) $\hat{R}$ is homogeneous of degree zero in (k,ω) for $k^2 + |\omega|^2 \geq 1$ and is a smooth function of all its variables and of the coefficient matrices A_j's and M,

(iii) $(y,\hat{R}y) \geq \delta|y|^2 - \varepsilon|u|^2$ for all vector y such that $My = u$ in Γ.

(iv) $\mathrm{Re}\ \hat{R}V \geq \delta k$ where $\delta > 0$ and $\varepsilon > 0$ are constants independent on k,ω and x.

Now we are in a position to prove the energy inequality (2.1). Multiplying the symbol $\hat{R}$ to (2.4) and taking the inner product with $\hat{y}$, we obtain

$$(2.5) \qquad (\hat{R}\,\frac{d\hat{y}}{dx_1},\ \hat{y})_\Omega = (\hat{R}V\hat{y},\hat{y})_\Omega$$

It is understood that the inner products in (2.5) are taken for the variables $(x_1,\omega_2,\ldots,\omega_m)$. From (iv), (2.5) becomes

$$(2.6) \qquad \mathrm{Re}\ (\hat{R}\,\frac{d\hat{y}}{dx_1},\ \hat{y})_\Omega \geq \delta k|\hat{y}|^2_\Omega.$$

On the other hand, using integration by parts with respect to x_1 and the fact that $\hat{y}|_{x_1=\infty} = 0$, we obtain

$$\mathrm{Re}\ (\hat{R}\,\frac{d\hat{y}}{dx_1},\ \hat{y})_\Omega = \mathrm{Re}[-(\hat{y},\hat{R}\hat{y})_\Gamma - (\hat{y},\ \frac{d\hat{R}}{dx_1}\,\hat{y})_\Omega - (\hat{y},\ \hat{R}\,\frac{d\hat{y}}{dx_1})_\Omega]$$

From (i), we have

$$(2.7) \qquad \mathrm{Re}\ (\hat{R}\,\frac{d\hat{y}}{dx_1},\ \hat{y})_\Omega = \frac{1}{2}\,\mathrm{Re}[-(\hat{y},\hat{R}\hat{y})_\Gamma - (\hat{y},\ \frac{d\hat{R}}{dx_1}\,\hat{y})_\Omega].$$

From (iii) and the fact that $(\hat{y}, \frac{d\hat{R}}{dx_1} \hat{y})_\Omega$ is of zero order, (2.7) becomes

$$(2.8) \qquad \mathrm{Re}\ (\hat{R}\ \frac{d\hat{y}}{dx_1}, \hat{y})_\Omega < -\frac{1}{2}\,\delta\,|\hat{y}|^2_\Gamma + \frac{1}{2}\,\varepsilon\,|\hat{u}|^2_\Gamma + c\ |\hat{y}|^2_\Omega$$

for some constant $c > 0$ independent on $\hat{y}$ and $\hat{u}$.

By (2.6) and (2.8), we obtain

$$(2.9) \qquad (\delta k-c)\ |\hat{y}|^2_\Omega + \frac{1}{2}\,\delta\,|\hat{y}|^2_\Gamma < \frac{1}{2}\,\varepsilon\,|\hat{u}|^2_\Gamma .$$

Since the Fourier transform is unitary, we may rewrite (2.9) as

$$(2.10) \qquad (\delta k-c)\ |y|^2_\Omega + \frac{1}{2}\,\delta\,|y|^2_\Gamma < \frac{1}{2}\,\varepsilon\,|u|^2_\Gamma .$$

By choosing k large enough so that $\delta k-c > 0$, we have

$$(2.11) \qquad |y|^2_\Omega + |y|^2_\Gamma < c\ |u|^2_\Gamma \text{ for some constant } c.$$

From (2.11), the energy inequality (2.1) can be easily derived.

Next we turn to the proof of existence.

First we notice that since the operator A generates a strongly continuous semigroup on $L^2(\Omega)$, there exists a real number r such that the set $\{\lambda \mid \mathrm{Re}\ \lambda > r\}$ is contained in the resolvent set of A. Hereafter, we choose and fix k large enough to be sure that (2.11) holds and $k > r$.
Then $(A-kI)$ has a bounded inverse on $L^2(\Omega)$. That is,

$$(2.12) \qquad \begin{aligned} &[A(x,\partial)-kI]y = f && \text{in } \Omega \\ &My = 0 && \text{in } \Gamma \end{aligned}$$

has a unique solution in $L^2(\Omega)$ for any f in $L^2(\Omega)$.

To continue with our proof, we first consider smooth boundary functions u in (1.16), i.e. $u \in H^1(\Gamma)$. It is well known that there exists a smooth extension $\bar{u}$ in $H^1(\Omega)$ of u.

Let $\tilde{u} = (\tilde{u}^-, \tilde{u}^+)$ where $\tilde{u}^- = \bar{u}$ and $\tilde{u}^+ = (0,\ldots,0)$.
Then $\tilde{u} \in H^1(\Omega;R^n)$ and $A(x,\partial)\tilde{u} \in L^2(\Omega)$.
Thus, from (2.12), it is easily seen that

$$(2.13) \qquad \begin{aligned} &[A(x,\partial)-kI]\tilde{y} = -A(x,\partial)\tilde{u} + k\tilde{u} && \text{in } \Omega \\ &M\tilde{y} = 0 && \text{in } \Gamma \end{aligned}$$

admits a unique solution $\tilde{y}$ in $L^2(\Omega)$.
From (1.5), $M\tilde{u} = u$ in Γ. Let $y = \tilde{y} + \tilde{u}$.

Then from (2.13), y is a solution to

$$(1.16) \qquad \begin{aligned} &A(x,\partial)y = ky && \text{in } \Omega \\ &My = u && \text{in } \Gamma. \end{aligned}$$

This implies the existence of solution to (1.16) for smooth boundary data u. By using a standard density argument, we extend our results to L^2-boundary data u as follows. For a given u in $L^2(\Gamma)$, we choose a sequence $\{u_n\}_n$ in $H^1(\Gamma)$ such that $u_n \to u$ in $L^2(\Gamma)$ as $n \to \infty$. By energy inequality (2.1), the solution to (1.16) is unique if it exists. Let y_n be the solution to (1.16) for the boundary data u_n; i.e.,

$$\begin{aligned} &A(x,\partial)y_n - ky_n = 0 && \text{in } \Omega \\ &My_n = u_n && \text{in } \Gamma. \end{aligned}$$

Then by (2.1), for any positive integers m, n,

$$(2.14)\qquad |y_n-y_m|_\Omega + |y_n-y_m|_\Gamma \leq c\ |u_n-u_m|_\Gamma.$$

Since $\{u_n\}_n$ is a Cauchy sequence in $L^2(\Gamma)$, from (2.14) we deduce that $\{y_n\}_n$ and $\{y_n|_\Gamma\}_n$ are Cauchy sequences in $L^2(\Omega)$ and $L^2(\Gamma)$ respectively. Thus, there exist the limits y in $L^2(\Omega)$ and y_Γ in $L^2(\Gamma)$ such that

$$(2.15)\qquad |y_n-y|_\Omega \to 0 \text{ and } |y_n-y_\Gamma|_\Gamma \to 0 \text{ as } n \to \infty.$$

From (2.15), we obtain $My_\Gamma = u$.
This means that y is a strong solution to (1.16).
Proof of theorem 2.1 is thus completed.

Remark. If the coefficients $A_j(x)$ in $A(x,\partial)$ are symmetric and block diagonal, then we can obtain (2.1) with the aid of a simple symmetrizer $\overline{R} = \begin{bmatrix} \varepsilon I_k & 0 \\ 0 & I_{n-k} \end{bmatrix}$ where I_k and I_{n-k} are k x k and (n-k) x (n-k) identity matrices respectively, and ε is a small positive number.
That is, $A_j = \begin{bmatrix} A_j^- & 0 \\ 0 & A_j^+ \end{bmatrix} = A_j^T$ where A_j^- are k x k symmetric matrix functions and A_j^+ are (n-k) x (n-k) symmetric matrix functions for $j=1,\dots,m$.
Suppose that y is a solution to (1.16).

Using integration by parts, it is easily seen that

$$(2.16)\qquad (\overline{R}A_1\partial_1 y,y)_\Omega = -\tfrac{1}{2}(\overline{R}A_1 y,y)_\Gamma - \tfrac{1}{2}(\overline{R}y,(\partial_1 A_1)y)_\Omega$$

and $(\overline{R}A_j\partial_j y,y)_\Omega = -\frac{1}{2}(\overline{R}y,(\partial_j A_j)y)_\Omega$ for $j \geq 2$.

Since $(\bar{R}A(x,\partial)y - \bar{R}ky,y)_\Omega = 0$, we have, from (2.16), that

$$(2.17) \quad (\bar{R}A_1y,y)_\Gamma + \sum_{j=1}^{m} (\bar{R}y,(\partial_jA_j)y)_\Omega + 2k(\bar{R}y,y)_\Omega = 0.$$

Boundary condition $My = u$ on Γ and (2.17) imply

$$(2.18) \quad \varepsilon(A_1^-u,u)_\Gamma - 2\varepsilon(A_1^-u,Ny^+)_\Gamma + \varepsilon(A_1^-Ny^+,Ny^+)_\Gamma$$

$$+ (A_1^+y^+,y^+)_\Gamma + 2k(\bar{R}y,y)_\Omega + \sum_{j=1}^{m} (\bar{R}y,(\partial_jA_j)y)_\Omega = 0.$$

Using the fact that $A_1^- < 0$, $A_1^+ > 0$, and that A_j, N are constant for $|x|$ large, we obtain that with $\varepsilon > 0$ small and $k > 0$ large,

$$(2.19) \quad |y|_\Omega^2 + |y|_\Gamma^2 \leq c\,|u|_\Gamma^2 \text{ for a constant } c > 0.$$

This completes the proof of (2.1) for this special case.

In the sequel, we need a representation of the trace operator in terms of the adjoint of the Dirichlet map D.

This can be done as follows.
Let A^* be the formal adjoint of $A(x,\partial)$.
By simple computation, we have

$$(2.20) \quad A^*y = -\sum_{j=1}^{m} A_j^T(x)\partial_jy - \sum_{j=1}^{m} \partial_j(A_j^T(x)y) \text{ for } y \in D(A^*)$$

where the domain $D(A^*) = \{y \mid A^*y \in L^2(\Omega),\ M^*y|_\Gamma = 0\}$, and the adjoint boundary operator M^* is given by

(2.21) $M^* = [-(A_1^+)^{-1} N^T A_1^- \vdots I_{n-k}]$.

Lemma 2.2. The operator $D^*(A^*-kI)$ is given by

$$D^*(A^*-kI)y = A_1^- \, y|_\Gamma^- \qquad \text{for } y \in D(A^*).$$

Proof. For $y \in D(A^*)$ and $u \in L^2(\Gamma)$, by Green's formula, we have

(2.22) $(A^*y, Du)_\Omega = (y, A(x,\partial)Du)_\Omega + (A_1 y, Du)_\Gamma$.

Thus, by (1.16), for $y \in D(A^*)$ and $u \in L^2(\Gamma)$, (2.22) becomes

(2.23) $(A^*y, Du)_\Omega = (y, kDu)_\Omega + (A_1 y, Du)_\Gamma$.

Since $y \in D(A^*)$, $M^*y = 0$ on Γ. From (2.21), we have

(2.24) $-(A_1^+)^{-1} N^T A_1^- y^- + y^+ = 0$ on Γ.

On the other hand, from (1.16), $MDu = u$ on Γ, i.e., $(Du)^- + N(Du)^+ = u$ on Γ.
Thus, from (2.23) and (2.24), we obtain

(2.25) $((A^*-kI)y, Du)_\Omega = (A_1^- y^-, u)_\Gamma$ for all u in $L^2(\Gamma)$.

This completes the proof of the lemma.

Remark. From Lemma 2.2, it is clear that the operator $D^*(A^*-kI)$ is densely defined; i.e., $D(A^*)$ is contained in the domain of $D^*(A^*-kI)$. Thus, it has the adjoint.

2.2 Input-Solution Formula for Hyperbolic Systems

In this section, we give a semigroup representation for the solution to (1.11); i.e., we give the proof of theorem 2.2. We recall the operator L defined in section 1.4,

$$(1.17) \qquad (Lu)(t) = (A-kI) \int_0^t S(t-s)Du(s)ds, \ 0 < t < T$$

for u in the domain of L.

We first observe that L is closed. This can be seen [7] by viewing L as a product of the operator (A-kI) which has a bounded inverse, with the closed operator $\int_0^t S(t-s)Du(s)ds$ which is actually bounded.

We next show that L is densely defined on $L^2(\Sigma)$. To do this, we compute $A \int_0^t S(t-s)f(s)ds$ for $f \in H^1([0,T]; D(A))$. Then we obtain, after integrating by parts,

$$(2.26) \qquad A \int_0^t S(t-s)f(s)ds =$$

$$\int_0^t S(t-s) \frac{d}{ds} f(s)ds + S(t)f(0)-f(t).$$

The right hand side of (2.26) can be extended to all f in $H^1([0,T]; L^2(\Omega))$. Thus,

$$(2.27) \qquad A \int_0^t S(t-s)Du(s)ds = \int_0^t S(t-s)D \frac{d}{ds} u(s)ds$$

$$+ S(t)Du(0)-Du(t)$$

is well defined for $u \in H^1([0,T]; L^2(\Gamma))$. Consequently, L is densely defined on $L^2(\Sigma)$.

In order to prove theorem 2.2, we need the following preliminary result.

Lemma 2.3. The operator $D^*A^*S^*(\cdot)$ is bounded from $L^2(\Omega)$ into $L^2(\Sigma)$. That is,

$$\int_0^T |D^*A^*S^*(t)x|_\Gamma^2 \, dt \leq c \, |x|_\Omega^2 \text{ for } x \in L^2(\Omega)$$

where c is a constant.

Proof. We consider the following mixed adjoint problem backward in time: for $x \in L^2(\Omega)$ and $f \in L^2(Q)$,

$$\begin{aligned} (2.28) \qquad & \partial_t z = -A^*z - f && \text{in } Q \\ & M^*z = 0 && \text{in } \Sigma \\ & Z(T) = x && \text{in } \Omega \end{aligned}$$

It is well-known [15] that the mixed problem (1.3) satisfies the Kreiss' condition if and only if its adjoint problem satisfies the Kreiss' condition with time traveling backward. Thus, Rauch's results can be applied to (2.28). By theorem 1.1, the following estimate holds:

$$(2.29) \qquad |z|_Q + |z|_\Sigma \leq c[|f|_Q + |x|_\Omega] \text{ for a constant c.}$$

On the other hand, the solution z can be expressed in terms of the adjoint semigroup $S^*(t)$ as

$$(2.30) \qquad z(t) = S^*(T-t)x + \int_t^T S^*(s-t)f(s)ds, \ 0 \leq t \leq T.$$

Since, by theorem 1.1, z has a well-defined trace on Γ and

$M^*z|_\Gamma = 0$, lemma 2.2 can be applied to (2.30). That is, $D^*(A^*-kI)z(t) = A_1^-(Z(t)|_\Gamma)^-$ for a.e. t. Thus,

$$(2.31) \qquad |D^*(A^*-kI)z|_\Sigma = |A_1^-(z|_\Gamma)^-|_\Sigma \leq c\ |z|_\Gamma|_\Sigma$$

for a constant $c > 0$, since A_1^- is constant outside a compact set. From (2.29), (2.30) and (2.31), we obtain

$$(2.32) \qquad |D^*(A^*-kI)S^*(T-\cdot)x + D^*(A^*-kI) \int_{(\cdot)}^T S^*(s-\cdot)f(s)ds|_\Sigma \leq c\ [|f|_Q + |x|_\Omega]$$

for $f \in L^2(Q)$ and $x \in L^2(\Omega)$.

By taking $f = 0$ in (2.32), we have

$$(2.33) \qquad |D^*(A^*-kI)S^*(T-\cdot)x|_\Sigma \leq c\ |x|_\Omega \text{ for } x \in L_2(\Omega).$$

Since $kD^*S^*(T-\cdot)$ is bounded from $L^2(\Omega)$ into $L^2(\Sigma)$, from (2.33), so is $D^*A^*S^*(T-\cdot)$. Thus, lemma 2.3 is proved.

Now we continue with the proof of theorem 2.2.

Proof of Theorem 2.2 (i). By taking $x = 0$ in (2.32), we obtain

$$(2.34) \qquad |D^*(A^*-kI) \int_{(\cdot)}^T S^*(s-\cdot)f(s)ds|_\Sigma \leq c\ |f|_Q \text{ for } f \in L^2(Q).$$

For a moment, let us define an operator $\bar{L}$ as

$$(2.35)\qquad (\overline{L}f)(t) = D^*(A^*-kI)\int_t^T S^*(s-t)f(s)ds$$

$$\text{for } f \in L^2(Q).$$

Then by (2.34), $\overline{L}$ is a bounded linear operator from $L^2(Q)$ into $L^2(\Sigma)$. It is easily seen that the adjoint of $\overline{L}$ as an operator from $L^2(Q)$ into $L^2(\Sigma)$ coincides with the operator L defined in (1.17). Since L is closed and densely defined, $L^* = \overline{L}$; i.e., $\overline{L}$ is the adjoint of L.

Thus, L is bounded from $L^2(\Sigma)$ into $L^2(Q)$.

Next we prove that L is bounded from $L^2(\Sigma)$ into $L^\infty([0,T];\ L^2(\Omega))$.

To do this, for $u \in L^2(\Sigma)$ and $f \in L^1([0,T];\ L^2(\Omega))$, we compute $(Lu,f)_Q$. By lemma 2.3 and Schwarz's inequality,

$$
\begin{aligned}
(2.36)\quad (Lu,f)_Q &= \int_0^T ((Lu)(t),f(t))_\Omega dt \\
&= \int_0^T \int_0^t (u(s),D^*(A^*-kI)S^*(t-s)f(t))_\Gamma ds dt \\
&\leq \int_0^T [\int_0^t |u(s)|_\Gamma^2 ds]^{\frac{1}{2}} [\int_0^t |D^*(A^*-kI)S^*(t-s)f(t)|_\Gamma^2 ds]^{\frac{1}{2}} dt \\
&\leq \int_0^T |u|_\Sigma [\int_0^t |D^*(A^*-kI)S^*(s)f(t)|_\Gamma^2 ds]^{\frac{1}{2}} dt \\
&\leq |u|_\Sigma \int_0^T |D^*(A^*-kI)S^*(\cdot)f(t)|_\Sigma dt \\
&\leq |u|_\Sigma \int_0^T c\ |f(t)|_\Omega dt \\
&= c\ |u|_\Sigma\ |f|_{L^1([0,T];\ L^2(\Omega))} \quad \text{for a constant } c > 0.
\end{aligned}
$$

That is, for $u \in L^2(\Sigma)$ and $f \in L^1([0,T];\ L^2(\Omega))$, $(Lu,f)_Q \leq c\ |u|_\Sigma\ |f|_{L^1([0,T];\ L^2(\Omega))}$. In fact, Lu is in the dual space of $L^1([0,T];\ L^2(\Omega))$, which in turn implies that Lu is in $L^\infty([0,T];\ L^2(\Omega))$ for each $u \in L^2(\Sigma)$.

In order to prove that Lu is in $C([0,T];\ L_2(\Omega))$ for $u \in L^2(\Sigma)$, we use a density argument.

From (2.27), for $u \in H^1([0,T];\ L_2(\Gamma))$, we have

$$(2.37) \quad (Lu)(t) = \int_0^t S(t-s)D\frac{d}{ds}u(s)ds + S(t)Du(0) - Du(t) - k\int_0^t S(t-s)Du(s)ds.$$

It is easily seen that Lu is in $C([0,T];\ L^2(\Omega))$ for $u \in H^1([0,T];\ L^2(\Gamma))$.

Since $H^1([0,T];\ L^2(\Gamma))$ is dense in $L^2(\Sigma)$, for any u in $L^2(\Sigma)$, there exists a sequence $\{u_n\}_n$ in $H^1([0,T];\ L^2(\Gamma))$ such that $|u_n-u|_\Sigma \to 0$ as $n \to \infty$.

For any positive integers m,n,

$$|Lu_m-Lu_n|_{C([0,T];\ L^2(\Omega))} = |Lu_m-Lu_n|_{L^\infty([0,T];\ L^2(\Omega))}$$

$$\leq c\ |u_m-u_n|_\Sigma \text{ for some constant } c > 0.$$

Hence $\{Lu_n\}_n$ is a Cauchy sequence in $C([0,T];\ L^2(\Omega))$.

By completeness of $C([0,T];\ L^2(\Omega))$, there exists the limit y in $C([0,T];\ L^2(\Omega))$ such that

$$|Lu_m-y|_{C([0,T];\ L^2(\Omega))} \to 0 \text{ as } n \to \infty.$$

On the other hand, $\{Lu_n\}_n$ converges to Lu in $L^\infty([0,T];\ L^2(\Omega))$ since L is bounded from $L^2(\Sigma)$ into $L^\infty([0,T];\ L^2(\Omega))$.

Thus, $Lu = y$ which completes the proof of part (i).

Proof of theorem 2.2 (ii). We first consider a smooth control $u \in H^1(\Sigma)$.

Suppose that y is the solution to (1.11) with $u \in H^1(\Sigma)$. Let $\bar{y} = y - Du$. Then, it is clear that $\bar{y}$ is the solution to

$$(2.38) \qquad \begin{aligned} &\partial_t \bar{y} = A(x,\partial)\bar{y} - D\frac{d}{dt}u + kDu && \text{in } Q \\ &M\bar{y} = 0 && \text{in } \Sigma \\ &\bar{y}(0) = y_o - Du(0) && \text{in } \Omega. \end{aligned}$$

That is,

$$(2.39) \qquad \begin{aligned} &\partial_t \bar{y} = A\bar{y} - D\frac{d}{dt}u + kDu && \text{in } Q \\ &\bar{y}(0) = y_o - Du(0). && \text{in } \Omega \end{aligned}$$

Using the variation of parameter formula, we obtain, from (2.39), that

$$(2.40) \qquad \begin{aligned} y(t) &= S(t)[y_o - du(0)] \\ &\quad - \int_o^t S(t-s)[D\frac{d}{ds}u(s) - kDu(s)]ds \\ &= S(t)y_o - S(t)Du(0) - \int_o^t S(t-s)\,D\frac{d}{ds}u(s)ds \\ &\quad + k\int_o^t S(t-s)Du(s)ds. \end{aligned}$$

Since $y(t) = \bar{y}(t) + Du(t)$, from (2.27) and (2.40), we obtain, for $u \in H^1(\Sigma)$,

$$(2.41) \qquad y(t) = S(t)y_o - (A-kI)\int_o^t S(t-s)Du(s)ds.$$

That is, $y(t) = S(t)y_o - (Lu)(t)$ for $u \in H^1(\Sigma)$.

Since $H^1(\Sigma)$ is dense in $L^2(\Sigma)$ and L is bounded in $L^2(\Sigma)$, the formula (2.41) can be extended to all of $L^2(\Sigma)$ by density argument.

This completes the proof of theorem 2.2.

From theorem 2.2 (i), we have the following.

Corollary 2.2. For each $t \in [0,T]$, let

$L^t u = (Lu)(t)$, $u \in L^2(\Sigma)$.

Then L^t is a bounded linear operator from $L^2(\Sigma)$ into $L^2(\Omega)$ uniformly in t, $0 \leq t \leq T$.

CHAPTER III
CHARACTERIZATION OF OPTIMAL CONTROL AND REGULARITY

3.1 Characterization of Optimal Control

In this section, we first establish the existence and uniqueness of the optimal control for our control problem with the aid of theorem 2.2.

We then provide an explicit representation of the optimal control and the corresponding trajectory in terms of the intial data. This will be done by using the Lagrange multiplier method [14].

In order to prove the existence and uniqueness of the optimal control, we recall the following result from [12].

Lemma 3.1. Let W be a real Hilbert space. Suppose that $\pi(\cdot,\cdot)$ is a continuous symmetric and coercive bilinear form on W.
Then for any continuous linear functional ϕ on W, the functional cost $J(u) = \pi(u,u) + \phi(u)$ admits a unique element u^o in a closed convex subset W_o of W such that $J(u_o) < J(u)$ for any $u \in W_o$, $u \neq u_o$.

From lemma 3.1, we obtain the following lemma.

Lemma 3.2. Our control problem (C.P) admits a unique optimal control u^o in $L^2(\Sigma)$.

Proof. By using the semigroup representation in theorem 2.2, we may rewrite $J(u,y)$ as

$$(3.1) \quad J(u,y) = \frac{1}{2}\,|u|_\Sigma^2 + \frac{1}{2}\,(S(\cdot)y_o - Lu, FS(\cdot)y_o - FLu)_Q + \frac{1}{2}\,(S(T)y_o - L^Tu, GS(T)y_o - GL^Tu)_\Omega = \pi(u,u) + \phi(u) + c \text{ for } u \in L^2(\Sigma)$$

where $\pi(u,v) = \frac{1}{2}\,(u,v)_\Sigma + \frac{1}{2}\,(Lu,FLv)_Q + \frac{1}{2}\,(L^Tu,GL^Tv)_\Omega$,

$\phi(u) = (Lu,FS(\cdot)y_o)_Q + (L^Tu,GS(T)y_o)_\Omega$ and

$c = \frac{1}{2}\,(S(\cdot)y_o,FS(\cdot)y_o)_Q + \frac{1}{2}\,(S(T)y_o,GS(T)y_o)_\Omega$.

Then it is easily seen, from theorem 2.2 (i) and Corollary 2.2, that π and ϕ satisfy all the requirements of lemma 3.1.

This proves the lemma.

Our next task is to give an explicit representation of the optimal solution. This is done in the next theorem.

Theorem 3.1.

The optimal control u^o and corresponding trajectory y^o are given by the following formulae: for $y_o \in L^2(\Omega)$,

(i) $u^o = (I+L^*FL+L^{T*}GL^T)^{-1}\,[L^*FS(\cdot) + L^{T*}GS(T)]y_o$ in $L^2(\Sigma)$,

(ii) $y^o = S(\cdot)y_o - Lu^o$ in $C([0,T]; L^2(\Omega))$.

Proof. By theorem 2.2 (ii), we may reformulate our control problem as follows:

(3.2) Minimize the functional cost

$$J(u,y) = \frac{1}{2}\,|u|^2_\Sigma + \frac{1}{2}\,(y,Fy)_Q + \frac{1}{2}\,(S(T)y_o - L^T u, GS(T)y_o - GL^T u)_\Omega$$

subject to the constraints

$$(3.3) \qquad y - S(\cdot)y_o + Lu = 0 \text{ over } u \in L^2(\Sigma).$$

At this point, we introduce Lagrange functional defined by

$$(3.4) \qquad \mathcal{L}(u,y,z) = J(u,y) + (z, y - S(\cdot)y_o + Lu)_Q$$

for $u \in L^2(\Sigma)$, $y \in L^2(Q)$ and $z \in L^2(Q)$.

It is clear that the Lagrange functional $\mathcal{L}$ is continuously Fréchet differentiable with respect to u, y and z in the corresponding spaces $L^2(\Sigma)$ and $L^2(Q)$. Since the optimal control exists uniquely in $L^2(\Sigma)$, it can be determined as a critical point of the Lagrangean $\mathcal{L}$ [14]. That is, if let $\mathcal{L}_u$ and $\mathcal{L}_y$ be the Fréchet derivatives of $\mathcal{L}$ with repsect to u and y respectively, then the optimality conditions are given by

$$(3.5) \qquad \mathcal{L}_u(u^o,y^o,z^o) = 0 \text{ and } \mathcal{L}_y(u^o,y^o,z^o) = 0.$$

Thus, by simple computations, (3.5) is reduced to

$$(3.6) \qquad z^o + Fy^o = 0$$

$$u^o + L^* z^o - L^{T*} GS(T)y_o + L^{T*} GL^T u^o = 0.$$

From (3.6), we obtain

$$(3.7)\quad u^o = L^*Fy^o + L^{T*}G(S(T)y_o - L^Tu^o).$$

Since $y^o = S(\cdot)y_o - Lu^o$, (3.7) becomes

$$(3.8)\quad (I+L^*FL+L^{T*}GL^T)u^o = L^*FS(\cdot)y_o + L^{T*}GS(T)y_o.$$

We notice in (3.8) that since F and G are self-adjoint and positive-definite, so is the operator $L^*FL+L^{T*}GL^T$, as an operator on $L^2(\Sigma)$. Thus, the spectrum of $L^*FL+L^{T*}GL^T$ is contained in the nonnegative real axis [17]. Therefore, -1 is in the resolvent set of $L^*FL+L^{T*}GL^T$. This implies that $I+L^*FL+L^{T*}GL^T$ has a bounded inverse on $L^2(\Sigma)$. Since the right hand side of (3.8) is in $L^2(\Sigma)$, we obtain the first part of the theorem.

The second part (ii) follows from the first part (i) and theorem 2.2 (ii).

Thus, the theorem is proved.

3.2 Regularity of Optimal Controls

The purpose of this section is to study the regularity of optimal control (theorem 3.2). In the general case, without assuming smoothness on the operators F and G, we can only say that u^o is in $L^2(\Sigma)$. If we, however, impose very mild assumptions (compactness type) on F and G, it can be said that the optimal control u^o is, in fact, smoother than L^2.

This result, besides being important in its own right, has also important application to numerical analysis of the control problem where regularity of optimal control plays a crucial role in determining rate of convergence of approximation.

In order to prove theorem 3.2, we assume additionally that

(A.4) F and G are compact operators on $H^s(\Omega)$ for all $0 \leq s < \frac{1}{2}$.

We define the following spaces; for nonnegative integers s,

$$\overset{\circ}{H}{}^s(\Sigma) = \{u \mid u \in H^s(\Sigma),\ u|_{t=0} = 0\} \text{ and } \tag{3.9}$$

$$\overset{T}{H}{}^s(Q) = \{y \mid y \in H^s(Q),\ y|_{t=T} = 0\}.$$

Then we have the following lemma.

Lemma 3.3. With $s = 0$ or 1, the following operators are bounded:

(i) $L: \overset{\circ}{H}{}^s(\Sigma) \to H^s(Q)$

(ii) $L^T: \overset{\circ}{H}{}^s(\Sigma) \to H^s(\Omega)$

(iii) $S(t): H_0^s(\Omega) \to H^s(\Omega)$ uniformly in t

(iv) $S(\cdot): H_0^s(\Omega) \to H^s(Q)$

(v) $L^*: \overset{T}{H}{}^s(Q) \to H^s(\Sigma)$

(vi) $L^{T*}: H_0^s(\Omega) \to H^s(\Sigma)$.

Proof. We recall the mixed problem (1.11) and an adjoint problem:

$$\begin{aligned} &\partial_t y = A(x,\partial)y && \text{in } Q \\ &My = u && \text{in } \Sigma \\ &y(0) = y_o && \text{in } \Omega \end{aligned}$$

and

$$\begin{aligned} &\partial_t z = -A^*z-f && \text{in } Q \\ &M^*z = 0 && \text{in } \Sigma \\ &Z(t) = z_o && \text{in } \Omega \end{aligned} \tag{3.10}$$

for $y_o, z_o \in L^2(\Omega)$, $f \in L^2(Q)$ and $u \in L^2(\Sigma)$. Then, as in Chapter II, the solution y and z are given by the following semigroup formulae;

$$y(t) = S(t)y_o - (Lu)(t) \tag{3.11}$$

$$z(t) = S^*(T-t)z_o + \int_t^T S^*(s-t)f(s)ds. \tag{3.12}$$

We now apply the regularity theorem 1.2 to (3.11) and (3.12). Then by taking $y_o = 0$ in (3.11), we obtain (i) and (ii). Similarly, by taking $u = 0$ in (3.11), (iii) and (iv) are obtained.

Let $z_o = 0$ in (3.12). From theorem 1.2, $z(t) = \int_t^T S^*(s-t)f(s)ds$ has a well-defined trace on the boundary in $H^s(\Sigma)$ if $f \in \overset{T}{H}{}^s(Q)$. Thus, we obtain (v) by using lemma 2.2 and theorem 1.2 together with the fact that $A_1^-(x)$ is smooth on Γ and constant for $|x|$ large.

Similarly, by taking $f = 0$ in (3.12), (vi) can be obtained.

Proof of theorem 3.2. By interpolating between $s = 0$ and 1 [13], lemma 3.3 holds for all $0 < s < 1$, $s \neq \frac{1}{2}$. On the other hand, for smooth domain Ω, the following identifications hold [13]: for $0 < s < \frac{1}{2}$,

$$(3.13) \qquad \overset{\circ}{H}\ (\Sigma) = H^s(\Sigma)$$

$$\overset{T}{H}{}^s(Q) = H^s(Q)$$

$$H_o^s(\Omega) = H^s(\Omega).$$

By using (3.13), we may rewrite lemma 3.3 as follows. For $0 < s < \frac{1}{2}$, the following operators are bounded:

(i)' $L: H^s(\Sigma) \to H^s(Q)$

(ii)' $L^T: H^s(\Sigma) \to H^s(\Omega)$

(iii)' $S(t): H^s(\Omega) \to H^s(\Omega)$ uniformly in t

(iv)' $S(\cdot): H^s(\Omega) \to H^s(Q)$

(v)' $L^*: H^s(Q) \to H^s(\Sigma)$

(vi)' $L^{T*}: H^s(\Omega) \to H^s(\Sigma)$.

Hence, for $0 < s < \frac{1}{2}$, $L^*FL+L^{T*}GL^T$ is a bounded operator on $H^s(\Sigma)$. Since F and G are compact, so is the operator $L^*FL+L^{T*}GL^T$, being a sum of compositions of compact operators with bounded operators. Thus, in order to show that $I+L^*FL+L^{T*}GL^T$ has a bounded inverse on $H^s(\Sigma)$, by the

Fredholm alternative, it is sufficient to prove that -1 is not an eigenvalue of $L^*FL+L^{T*}GL^T$ on $H^s(\Sigma)$. On the other hand, we know in the previous section that -1 is not an eigenvalue of $L^*FL+L^{T*}GL^T$ on $L^2(\Sigma)$. Since $H^s(\Sigma)$ is contained in $L^2(\Sigma)$, -1 cannot be an eigenvalue of $L^*FL+L^{T*}GL^T$ on $H^s(\Sigma)$.

Our next step is to show that $L^*FS(\cdot)y_o+L^{T*}GS(T)y_o$ is in $H^s(\Sigma)$ for $y_o \in H^s(\Omega)$. This follows immediately from (iii)' - (vi)'.

Thus, $u^o = (I+L^*FL+L^{T*}GL^T)^{-1}[L^*FS(\cdot)y_o+L^{T*}GS(T)y_o]$ is in $H^s(\Sigma)$ for $y_o \in H^s(\Omega)$.

Part (ii) follows from part (i) and theorem 3.1 (ii).

Thus, the theorem is proved.

Remark. If $s > \frac{1}{2}$, then the identifications (3.13) do not hold. Thus, in view of theorem 1.2, the operator $L^*FL+L^{T*}GL^T$ is, in general, not bounded from $H^s(\Sigma)$ into itself.

In the sequel, we need also regularity of optimal control for regularized problem. More precisely, we have the following result.

Theorem 3.3.

Assume that (A.5) F and G map $L^2(\Omega)$ into $H^1_o(\Omega)$. Then $u^o \in H^1(\Sigma)$ for any initial data $y_o \in L^2(\Omega)$.

Proof. By (A.5), GL^Tu and $GS(T)y_o$ are in $H^1_o(\Omega)$ for any $u \in L^2(\Sigma)$ and $y_o \in L^2(\Omega)$.

From lemma 3.3 (vi), we deduce that

$$(3.14) \quad L^{T*}GL^{T}u \text{ and } L^{T*}GS(T)y_o \text{ are in } H^1(\Sigma)$$

$$\text{for any } u \in L^2(\Sigma) \text{ and } y_o \in L^2(\Omega).$$

Next we show that $L^*FS(\cdot)y_o$ and L^*FLu are in $H^1(\Sigma)$ for $y_o \in L^2(\Omega)$ and $u \in L^2(\Sigma)$. In fact, for $y_o \in D(A)$, we have

$$(3.15) \quad \frac{d}{dt} FS(t)y_o = FAS(t)y_o \in L^2([0,T];\ H^1(\Omega)).$$

By (A.5), the operator FA can be extended to be bounded on $L^2(\Omega)$ [1 p. 80]. Since D(A) is dense in $L^2(\Omega)$, the formula (3.15) can be extended to all of $L^2(\Omega)$. Consequently, we deduce that

$$(3.16) \quad FS(\cdot)y_o \in H^1(Q) \text{ for } y_o \in L^2(\Omega).$$

Let $f(s) = FS(s)y_o - FS(T)y_o$ for $y_o \in L^2(\Omega)$. Then $f \in \overset{T}{H}{}^1(Q)$ for $y_o \in L^2(\Omega)$. By lemma 3.3 (v), we have

$$(3.17) \quad L^*f \in H^1(\Sigma);\ \text{i.e., } L^*FS(\cdot)y_o - L^*FS(T)y_o \in H^1(\Sigma).$$

On the other hand, by theorem 1.2, $S^*(\cdot-t)FS(T)y_o$ has a well-defined trace on Γ in $H^1([t,T]\times\Gamma)$. In fact, by lemma 2.2, we obtain

$$(3.18) \quad [D^*(A^*-kI)S^*(\cdot-t)FS(T)y_o]\big|_\Gamma$$

$$= A_1^-([S^*(\cdot-t)FS(T)y_o]\big|_\Gamma)^- \text{ which is in } H^1([t,T]\times\Gamma).$$

From (3.18), for $0 < t < T$, we have

$$(3.19) \quad (L^*FS(T)y_o)(t) = D^*(A^*-kI)\int_t^T S^*(s-t)FS(T)y_o ds$$

$$= \int_t^T A_1^- ([S^*(s-t)FS(T)y_o]\big|_\Gamma)^- ds \text{ is in } H^1(\Gamma).$$

Moreover,

$$(L^*FS(T)y_o)(t) = D^* \int_t^T S^*(s-t)(A^*-kI)FS(T)y_o ds$$

$$= D^* \int_o^{T-t} S^*(s)(A^*-kI)FS(T)y_o ds$$

is absolutely continuous in t. Clearly, the derivative, $-D^*S^*(T-t)A^*FS(T)y_o$ a.e. t, belongs to $L^2(\Sigma)$. Therefore, by (3.19), we obtain that $L^*FS(T)y_o$ is in $H^1(\Sigma)$, and consequently from (3.17) that $L^*FS(\cdot)y_o$ is in $H^1(\Sigma)$. Hence, from (3.14), we have

$$(3.20) \quad L^*FS(\cdot)y_o + L^{T*}GS(T)y_o \in H^1(\Sigma) \text{ for } y_o \in L^2(\Omega).$$

Next we will show that $L^*FLu \in H^1(\Sigma)$ for $u \in L^2(\Sigma)$. We first show that $FLu \in H^1(Q)$ for $u \in L^2(\Sigma)$. Since $FLu \in C([0,T]; H_o^1(\Omega))$, it is enough to show that $\frac{d}{dt} FLu \in L^2(Q)$. For u in $C^1([0,T]; L^2(\Gamma))$, $\int_o^t S(t-s)Du(s)ds$ is continuously differentiable in t [3, p. 29]. Moreover, $\frac{d}{dt} \int_o^t S(t-s)Du(s)ds = Du(t) + A \int_o^t S(t-s)Du(s)ds$. Since $F(A-kI)$ is bounded, we obtain

$$(3.21) \quad \frac{d}{dt} (FLu)(t) = F(A-kI) \frac{d}{dt} \int_o^t S(t-s)Du(s)ds$$

$$= F(A-kI)Du(t) + F(A-kI)(Lu)(t)$$

$$+ F(A-kI)\, k \int_o^t S(t-s)Du(s)ds \text{ for } u \in C^1([0,T]; L^2(\Gamma)).$$

The right hand side of (3.21) is in $L^2(Q)$ for $u \in L^2(\Sigma)$.

Thus, by density argument, $\frac{d}{dt} (FLu)(t) \in L^2(Q)$ for $u \in L^2(\Sigma)$.

That is,

$$(3.22) \quad FLu \in H^1(Q) \text{ for } u \in L^2(\Sigma).$$

Now by lemma 3.3 (v) and (3.22), we obtain

$$(3.23) \quad L^*(FLu-FL^Tu) \in H^1(\Sigma) \text{ for } u \in L^2(\Sigma).$$

By the similar way as in (3.17) ~ (3.19), it is easily verified that (3.24) $L^*FLu \in H^1(\Sigma)$ for $u \in L^2(\Sigma)$.

Combining (3.14) with (3.24), we obtain

$$(3.25) \quad L^*FL+L^{T*}GL^T \text{ maps } L^2(\Sigma) \text{ into } H^1(\Sigma).$$

Since $I+L^*FL+L^{T*}GL^T$ has an inverse on $L^2(\Sigma)$, from (3.25) we conclude that its inverse $(I+L^*FL+L^{T*}GL^T)^{-1}$ maps $H^1(\Sigma)$ into itself. Thus, by (3.20), we obtain

$$u^o = (I+L^*FL+L^{T*}GL^T)^{-1}(L^*FS(\cdot)y_o + L^{T*}GS(T)y_o)$$

is in $H^1(\Sigma)$ for $y_o \in L^2(\Omega)$.

This completes the proof of the theorem.

CHAPTER IV
EVOLUTION OPERATOR AND RICCATI OPERATOR

4.1 Evolution Operator

In the previous chapter, it has been shown that the optimal control and corresponding trajectory are L^2-functions of time with values in $L^2(\Gamma)$ and $L^2(\Omega)$ respectively.

However, the expression for the optimal control in theorem 3.1 is given in terms of the initial data. This requires knowledge of the whole process starting from the initial time $t = 0$.

In order to obtain a feedback realization of the optimal control (pointwise in time), we need a representation of the optimal control as an operator acting on the trajectory at each time.

In order to accomplish this, we introduce an evolution operator Φ which describes the evolution of the control system.

Let s be an arbitrary time in $[0,T)$. We, henceforth, regard s as a new initial time for our control problem, with new initial data $x \in L^2(\Omega)$.

In fact, our new control problem over the time interval $[s,T]$ with initial data x is as follows.

$(C.P)_s$ Minimize the functional cost

$$J_s(u,y) = \frac{1}{2} |u|^2_{[s,T]x\Gamma} + \frac{1}{2} (y,Fy)_{[s,T]x\Omega} + \frac{1}{2} (y(T),Gy(T))_\Omega$$

over $u \in L^2([s,T]x\Gamma)$ subject to the mixed problem

$$(4.1) \qquad \begin{aligned} \partial_t y &= A(x,\partial)y && \text{in } [s,T]x\Omega \\ My &= u && \text{in } [s,T]x\Gamma \\ y(s) &= x && \text{in } \Omega. \end{aligned}$$

Then, by the same argument as in Chapter III for (C.P), it is clearly seen that the above problem $(C.P)_s$ admits a unique optimal control in $L^2([s,T]x\Gamma)$.

Let us denote by $u^o(\cdot,s;x)$ and $y^o(\cdot,s;x)$ the optimal control and corresponding trajectory for $(C.P)_s$ respectively.

In order to obtain explicit representation for $u^o(\cdot,s;x)$ and $y^o(\cdot,s;x)$, we define the following operators as in Chapter II:

$$(4.2) \qquad (L_s u)(t) = (A-kI) \int_s^t S(t-r)Du(r)dr \text{ for } u \in D(L_s),$$

$$(4.3) \qquad (L_s^* f)(t) = \begin{cases} (L^*f)(t), & s \leq t \leq T \\ 0 & , 0 \leq t < s \end{cases} \quad \text{for } f \in D(L_s^*).$$

For $0 \leq s \leq t \leq T$, we define the operator L_s^t as

$$L_s^t u = (L_s u)(t) \text{ for } u \in D(L_s^t).$$

Then, by the same argument as in Chapter II, we obtain the following lemma.

Lemma 4.1.

(i) The operator L_s is bounded from $L^2([s,T]x\Gamma)$ into $C([s,T]; L^2(\Omega))$ uniformly in $s \in [0,T]$.

(ii) For each $u \in L^2(\Sigma)$, $L_s^t u$ is continuous in both s and t, $0 \leq s \leq t \leq T$.

(iii) The solution y_s of (4.1) is given by

$y_s(t) = S(t-s)x - L_s^t u$ for $s \leq t \leq T$.

Proof. Part (i) and (ii) follow directly from theorem 2.2 and corollary 2.2.

For part (ii), it is, in view of theorem 2.2 (i), sufficient to show that $L_s^t u = L^t u - S(t-s)\, L^s u$ for $u \in L^2(\Sigma)$ and $0 \leq s \leq t \leq T$. In fact, for $u \in L^2(\Sigma)$, $0 \leq s \leq t \leq T$,

$$\begin{aligned} L_s^t u &= (A-kI) \int_s^t S(t-r)Du(r)dr \\ &= (A-kI) \int_o^t S(t-r)Du(r)dr - (A-kI) \int_o^s S(t-r)Du(r)dr \\ &= L^t u - (A-kI) \int_o^s S(t-r)Du(r)dr. \end{aligned}$$

We next show that $(A-kI) \int_o^s S(t-r)Du(r)dr = S(t-s)\, L^s u$ for each $0 \leq s \leq t \leq T$. For x in $D(A^*)$, we have

$$\begin{aligned} &((A-kI) \int_o^s S(t-r)Du(r)dr, x)_\Omega \\ &\qquad = (\int_o^s S(s-r)Du(r)dr,\ S^*(t-s)(A^*-kI)x)_\Omega \\ &\qquad = (\int_o^s S(s-r)Du(r)dr,\ (A^*-kI)S^*(t-s)x)_\Omega \\ &\qquad = (L^s u,\ S^*(t-s)x)_\Omega = (S(t-s)L^s u, x)_\Omega \end{aligned}$$

which can be extended to all of $L^2(\Omega)$. Thus, we obtain for $0 \leq s \leq t$,

$$(A-kI) \int_0^s S(t-r)Du(r)dr = S(t-s)L^s u.$$

This completes the proof of the lemma.

Now applying the procedures of Chapter III to $(C.P)_s$, we have

$$(4.5) \quad u^o(\cdot,s;x) = L_s^* F y^o(\cdot,s;x) + L_s^T{}^* G y^o(T,s;x),$$

$$(4.6) \quad y^o(\cdot,s;x) = S(\cdot-s)x - L_s u^o(\cdot,s;x).$$

We obtain the following lemma similar to theorem 3.1.

Lemma 4.2. The optimal control for $(C.P)_s$ is given by the following formula;

$$u^o(\cdot,s;x) = (I+L_s^* F L_s + L_s^T{}^* G L_s^T)^{-1} [L_s^* F S(\cdot-s)x + L_s^T{}^* G S(T-s)x].$$

From (4.6), we are in a position to define an evolution operator as

$$(4.7) \quad \Phi(t,s)x = y^o(t,s;x)$$

$$\text{for } 0 \leq s \leq t \leq T,\ x \in L^2(\Omega),$$

that is, $\Phi(t,s)x = S(t-s)x - (L_s u^o(\cdot,s;x))(t)$.
Then $\Phi(t,s)$ is the operator which describes the evolution of the optimal trajectories.

Some properties of Φ are collected below.

Lemma 4.3. The operator Φ satisfies

(i) $\Phi(t,t) = I$ identity operator for $0 \le t \le T$,

(ii) $\Phi(r,t)\Phi(t,s) = \Phi(r,s)$ for $0 \le s \le t \le r \le T$,

(iii) $\Phi(t,s)$ is strongly continuous in t from $L^2(\Omega)$ into $C([s,T];\, L^2(\Omega))$ for each $0 \le s < T$,

(iv) $\|\Phi(t,s)\|_{L^2(\Omega)\to L^2(\Omega)} \le c$, all $0 \le s \le t \le T$ for a constant c,

(v) $\Phi(t,s)$ is strongly continuous in s from $L^2(\Omega)$ into $C([0,t];\, L^2(\Omega))$ for each $0 < t \le T$.

Proof. Part (i) is obvious from (4.7). Part (ii) can be easily obtained from the uniqueness of optimal control and trajectory for a given data. Part (iii) follows from the fact that the semigroup is strongly continuous and the operator L_s is bounded from $L^2([s,T]\times\Gamma)$ into $C([s,T];\, L^2(\Omega))$. For part (iv), we first show that $|u^o(\cdot,s;x)|_{[s,T]\times\Gamma} \le c\, |x|_\Omega$ uniformly in s for all $x \in L^2(\Omega)$.

To do this, we use lemma 4.2. Since $L_s^*FL_s+L_s^{T*}GL_s^T$ is a positive-definite operator on $L^2([s,T]\times\Gamma)$ for each s, we have

$$(4.8) \qquad \|I+L_s^*FL_s+L_s^{T*}GL_s^T\|_{L^2([s,T]\times\Gamma)\,\to\,\text{itself}} \ge 1.$$

Thus,

$$(4.9) \qquad \|(I+L_s^*FL_s+L_s^{T*}GL_s^T)^{-1}\|_{L^2([s,T]\times\Gamma)\,\to\,\text{itself}} \le 1$$

for all $0 \le s \le T$ [17].

Lemma 4.2 and (4.9) yield

$$(4.10)\quad |u^{o}(\cdot,s;x)|_{[s,T]\times\Gamma} \leq |L_s^{*}FS(\cdot-s)x + L_s^{T*}GS(T-s)x|_{[s,T]\times\Gamma}$$

for $x \in L^2(\Omega)$ and $0 \leq s \leq T$.

On the other hand, in view of lemma 4.1 the operator norms of L_s^{*} and L_s^{T*} are uniformly bounded in s; i.e., for $0 \leq s < T$,

$$(4.11)\quad \|L_s^{*}\|_{L^2([s,T]\times\Omega)\to L^2([s,T]\times\Gamma)} \leq \|L^{*}\|_{L^2(Q)\to L^2(\Sigma)},$$

$$\|L_s^{T*}\|_{L^2(\Omega)\to L^2([s,T]\times\Gamma)} \leq \|L^{T*}\|_{L^2(\Omega)\to L^2(\Sigma)}.$$

By (4.10) and (4.11), together with the boundedness of $FS(\cdot-s)$ and $GS(T-s)$ uniformly in s, we obtain

$$(4.12)\quad |u^{o}(\cdot,s;x)|_{[s,T]\times\Gamma} \leq c\ |x|_{\Omega} \text{ for } x \in L^2(\Omega)$$

and $0 \leq s < T$. From (4.2), it follows that for $0 \leq s < T$,

$$(4.13)\quad \|L_s\|_{L^2([s,T]\times\Gamma)\to C([s,T];\ L^2(\Omega))} \leq \|L\|_{L^2(\Sigma)\to C([0,T];\ L^2(\Omega))}.$$

Thus, from (4.12) and (4.13), we have

$$(4.14)\quad |(L_s u^{o}(\cdot,s;x))(t)|_{\Omega} \leq c\ |x|_{\Omega}$$

for $0 \leq s \leq t \leq T$, $x \in L^2(\Omega)$, where c is a constant independent on s, t and x. Since $|S(t-s)x|_\Omega \leq c\ |x|_\Omega$ for $0 \leq s \leq t \leq T$, $x \in L^2(\Omega)$ and some constant c, from (4.7) and (4.14) we obtain (iv). For part (v), we recall that strong continuity (iii) in the first variable t together with uniform boundedness (iv) yields strong continuity in the second variable s for $\Phi(t,s)$ as in [1].

This completes the proof of the lemma.

4.2 Riccati Operator and Feedback Representation of Optimal Control

In this section, we introduce an operator (so called a Riccati operator) on $L^2(\Omega)$ which synthesize the optimal control in a feedback form. Then we show that the above operator satisfies a Riccati integral equation.

Let us define an operator P(t) on $L^2(\Omega)$ as follows:

$$(4.15) \qquad P(t)x = \int_t^T S^*(s-t)F\Phi(s,t)x\,ds + S^*(T-t)G\Phi(T,t)x$$

for $x \in D(P(t)) \subset L^2(\Omega)$, and $0 \leq t \leq T$.

It is easily seen, from lemma 4.2, that P(t) is a bounded linear operator on $L^2(\Omega)$ for each t. Moreover, P(t) is strongly continuous from $L^2(\Omega)$ into $C([0,T]; L^2(\Omega))$ with uniform bound; i.e., $\|P(t)\|_{L^2(\Omega)\to L^2(\Omega)} \leq c$ for all $t \in [0,T]$.

Now we are in a position to prove that the above operator P(t) satisfies theorem 4.1.

Proof of theorem 4.1 (i). For any $x \in L^2(\Omega)$ and $0 \leq s \leq t \leq T$, from (4.5), we obtain

$$(4.16)\quad u^o(t,s;x) = (L_s^* F y^o(\cdot,s;x))(t) + (L_s^{T*} G y^o(T,s;x))(t)$$

$$= D^*(A^*-kI) \int_s^T S^*(r-t) F y^o(r,s;x) dr$$

$$+ D^*(A^*-kI) S^*(T-t) G y^o(T,s;x).$$

Since $y^o(r,s;x) = \Phi(r,t)\Phi(t,s)x$, (4.16) becomes

$$(4.17)\quad u^o(t,s;x) = D^*(A^*-kI) \int_t^T S^*(r-t) F \Phi(r,t)\Phi(t,s) x dr$$

$$+ D^*(A^*-kI) S^*(T-t) G \Phi(T,t)\Phi(t,s) x$$

$$= D^*(A^*-kI) P(t) \Phi(t,s) x.$$

Thus, by taking $t = s$ and $x = y^o(t) = \Phi(t,0)y_o$, we obtain, from (4.17), that

$$u^o(t) = D^*(A^*-kI)P(t)y^o(t) = BP(t)y^o(t),$$

where $B = D^*(A^*-kI)$ is given in lemma 2.2.

This completes the proof of theorem 4.1 (i).

We first prove the Riccati integral equation in theorem 4.1 (ii) under an additional smoothness assumptions on F and G in this section. Then, later in Chapter V, we will prove the Riccati equation without smoothness assumptions on F and G, by using limit process.

Thus, for the time being, let us assume

(A.6) F and G map $L^2(\Omega)$ into the domain of A^*.

Proof of theorem 4.1 (ii) under (A.6). We recall, from (4.7) and (4.17), that for $x \in L^2(\Omega)$ and $0 \leq s \leq t \leq T$,

$$\Phi(t,s)x = S(t-s)x - (L_s BP(\cdot)\Phi(\cdot,s)x)(t) \tag{4.18}$$

$$= S(t-s)x - (A-kI) \int_s^t S(t-r)DBP(r)\Phi(r,s)x\,dr.$$

From (4.15), we obtain that for $x, y \in L^2(\Omega)$ and $0 \leq t \leq T$,

$$(P(t)x,y)_\Omega = \int_t^T (\Phi(s,t)x, FS(s-t)y)_\Omega ds + (\Phi(T,t)x, GS(T-t)y)_\Omega. \tag{4.19}$$

Substituting (4.18) into (4.19), (4.19) becomes

$$(P(t)x,y)_\Omega = \int_t^T (\Phi(s,t)x, F\Phi(s,t)y)_\Omega ds + (\Phi(T,t)x, G\Phi(T,t)y)_\Omega + I_1 + I_2 \tag{4.20}$$

where

$$I_1 = \int_t^T (F\Phi(s,t)x,\ (A-kI) \int_t^s S(s-r)DBP(r)\Phi(r,t)y\,dr)_\Omega ds$$

and $I_2 = (G\Phi(T,t)x,\ (A-kI) \int_t^T S(T-r)DBP(r)\Phi(r,t)y\,dr)_\Omega$.

Using the fact that $(A^*-kI)F$ and $(A^*-kI)G$ are bounded on $L^2(\Omega)$ by (A.6), I_1 and I_2 become

$$I_1 = \int_t^T ((A^*-kI)F\Phi(s,t)x,\ \int_t^s S(s-r)DBP(r)\Phi(r,t)y\,dr)_\Omega ds$$

and $I_2 = ((A^*-kI)G\Phi(T,t)x,\ \int_t^T S(T-r)DBP(r)\Phi(r,t)y\,dr)_\Omega$ where due to (A.6), both integrands in I_1 and I_2 are absolutely integrable.

Thus, we can use Fubini's theorem to obtain

$$(4.21)\quad I_1 = \int_t^T \int_t^s ((A^*-kI)F\Phi(s,t)x, S(s-r)DBP(r)\Phi(r,t)y)_\Gamma drds$$

$$= \int_t^T \int_t^s (BS^*(s-r)F\Phi(s,t)x, BP(r)\Phi(r,t)y)_\Gamma drds$$

and

$$I_2 = \int_t^T ((A^*-kI)G\Phi(T,t)x, S(T-r)DBP(r)\Phi(r,t)y)_\Omega dr$$

$$= \int_t^T (BS^*(T-r)G\Phi(T,t)x, BP(r)\Phi(r,t)y)_\Gamma dr.$$

After changing the order of integration in (4.21), I_1 becomes

$$I_1 = \int_t^T \int_r^T (BS^*(s-r)F\Phi(s,t)x, BP(r)\Phi(r,t)y)_\Gamma dsdr.$$

Using Fubini's theorem again, we obtain

$$I_1 = \int_t^T (B \int_r^T S^*(s-r)F\Phi(s,r)\Phi(r,t)xds, BP(r)\Phi(r,t)y)_\Gamma dr.$$

Thus, we have, from (4,15), that

$$(4.22)\quad I_1 + I_2 = \int_t^T (B \int_r^T S^*(s-r)F\Phi(s,r)\Phi(r,t)xds$$

$$+ BS^*(T-r)G\Phi(T,r)\Phi(r,t)x, BP(r)\Phi(r,t)y)_\Gamma dr$$

$$= \int_t^T (BP(r)\Phi(r,t)x, BP(r)\Phi(r,t)y)_\Gamma dr.$$

From (4.20) and (4.22), it is readily seen that P(t) satisfies the first Riccati integral equation.

This completes the proof of theorem 4.1 (ii) under (A.6).

Notice that we use the assumption (A.6) only to justify change of the order of integrations in I_1 and I_2.

Remark. From the first Riccati integral equation, it follows immediately that under (A.6), our Riccati operator P(t) is self-adjoint and positive-definite.

CHAPTER V
RICCATI EQUATIONS

5.1 Derivation of Riccati Equations

The Riccati integral equation stated in theorem 4.1 is expressed in terms of the feedback evolution operator Φ. Another representation (so called the second Riccati integral equation) of the Riccati operator $P(t)$ in terms of only the original semigroup $S(t)$ is available under assumption (A.6). Also, a Riccati differential equation for $P(t)$ is derived under assumption (A.6).

The purpose of this section is the proof of theorem 5.1. In order to prove the theorem, as a preliminary step we need some results on traces of the Riccati operator $P(t)$ as well as on regularity of the evolution operator Φ.

Remark. The presence of the constant k in the operator B does not affect regularity of $BP(t)$ and Φ. Thus, from now on, without loss of generality, we assume $k = 0$.

Lemma 5.1. Under assumption (A.6),

(i) $BP(\cdot)x$ is in $C([0,T]; L^2(\Gamma))$ for $x \in L^2(\Omega)$,

(ii) $BP(\cdot)S(\cdot)B^*u$ is in $L^2(\Sigma)$ for $u \in L^2(\Gamma)$.

Proof of part (i). From the assumption (A.6), we deduce that $BS^*(s-t)F$ and $BS^*(T-t)G$ are bounded from $L^2(\Omega)$ into $L^2(\Gamma)$ uniformly in s and t, $0 \leq t \leq s \leq T$. Thus, for $x \in L^2(\Omega)$, $0 \leq t \leq T$, we have

$$(5.1)\quad |BP(t)x|_\Gamma = \left|\int_t^T BS^*(s-t)F\Phi(s,t)x\,ds + BS^*(T-t)G\Phi(T,t)x\right|_\Gamma$$

$$\leq \int_t^T |BS^*(s-t)F\Phi(s,t)x|_\Gamma ds + |BS^*(T-t)G\Phi(T,t)x|_\Gamma$$

$$\leq \int_t^T \|BS^*(s-t)F\|\,\|\Phi(s,t)\|\,|x|_\Omega ds + \|BS^*(T-t)G\|\,\|\Phi(T,t)\|\,|x|_\Omega .$$

Combining (5.1) with lemma 4.3 (iv), we obtain

$$(5.2)\quad |BP(t)x|_\Gamma \leq c\,|x|_\Omega \text{ uniformly in } t \in [0,T]$$

$$\text{for } x \in L^2(\Omega).$$

The continuity of $BP(t)x$ in t follows from strong continuity of S^* and Φ. Thus, part (i) is proved.

Proof of part (ii). We recall, from (4.7) and lemma 4.1, that

$$(5.3)\quad \Phi(s,t)x = S(s-t)x - L_t^s(I+L_t^*FL_t+L_t^{T*}GL_t^T)^{-1}$$

$$[L_t^*FS(\cdot-t)x + L_t^{T*}GS(T-t)x]$$

$$\text{for } x \in L^2(\Omega),\ 0 \leq t \leq s \leq T.$$

We next show that $B\int_t^T S^*(s-t)F\Phi(s,t)S(t)B^*u\,ds$ and $BS^*(T-t)G\Phi(T,t)S(t)B^*u$ are well-defined in $L^2(\Sigma)$ for $u \in L^2(\Gamma)$. By (5.3), we obtain that for $u \in L^2(\Gamma)$, $0 \leq t \leq T$,

$$(5.4)\quad B\int_t^T S^*(s-t)F\Phi(s,t)S(t)B^*u\,ds = B\int_t^T S^*(s-t)FS(s)B^*u\,ds$$

$$- B\int_t^T S^*(s-t)FL_t^s(I+L_t^*FL_t+L_t^{T*}GL_t^T)^{-1}[L_t^*FS(\cdot)B^*u+L_t^{T*}GS(T)B^*u]ds.$$

In view of (A.6) and boundedness of L^*, the first integral on the right hand side of (5.4) is in $L^2(\Sigma)$. From (A.6), it is easily seen that $FS(\cdot)B^*u \in L^2(Q)$ and $GS(T)B^*u \in L^2(\Omega)$ for $u \in L^2(\Gamma)$. The second integral of the right hand side of (5.4) becomes

$$D^*\int_t^T S^*(s-t)A^*FL_t^s(I+L_t^*FL_t+L_t^{T*}GL_t^T)^{-1}[L_t^*FS(\cdot)B^*u+L_t^{T*}GS(T)B^*u]ds$$

which is in $L^2(\Sigma)$ since the integrand is uniformly bounded in $t \in [0,T]$. Thus, from (5.4), we have

$B\int_t^T S^*(s-t)F\Phi(s,t)S(t)B^*u\,ds \in L^2(\Sigma)$ for $u \in L^2(\Gamma)$.

Also, from (5.3), we have that for $u \in L^2(\Gamma)$, $0 \leq t \leq T$,

$$(5.5)\quad BS^*(T-t)G\Phi(T,t)S(t)B^*u = BS^*(T-t)GS(T)B^*u$$

$$- BS^*(T-t)GL_t^T(I+L_t^*FL_t+L_t^{T*}GL_t^T)^{-1}[L_t^*FS(\cdot)B^*u+L_t^{T*}GS(T)B^*u].$$

By the same argument as before, we obtain that

$$BS^*(T-t)GS(T)B^*u \in L^2(\Sigma) \text{ for } u \in L^2(\Gamma).$$

The second part of the right hand side of (5.5) is also in $L^2(\Sigma)$ since it is bounded uniformly in $t \in [0,T]$. Thus, from (5.5), $BS^*(T-\cdot)G\Phi(T,\cdot)S(\cdot)B^*u \in L^2(\Sigma)$ for $u \in L^2(\Gamma)$. Consequently, we conclude that, from the definition of $P(t)$, $BP(\cdot)S(\cdot)B^*u \in L^2(\Sigma)$ for $u \in L^2(\Gamma)$.

This completes the proof of the lemma.

We have the following regularity for Φ.

Lemma 5.2 Differentiability of Φ in $[D(A^*)]'$. For $x \in L^2(\Omega)$ and a.e. $t \in [0,T]$, we have $\frac{d}{dt}\Phi(s,t)x = -\Phi(s,t)A[I-DBP(t)]x$ where the equality is understood in the dual space $[D(A^*)]'$.

Proof. We first show that for $x \in L^2(\Omega)$, $0 \leq t \leq s \leq T$,

(5.6) $\Phi(s,t)Ax \in [D(A^*)]'$ uniformly in t and s.

For $x \in D(A)$, $0 \leq t \leq s \leq T$, from (4.7), we have

$$(5.7) \quad \Phi(s,t)Ax = S(s-t)Ax$$

$$- L_t^s(I+L_t^*FL_t+L_t^{T*}GL_t^T)^{-1}[L_t^*FS(\cdot-t)Ax + L_t^{T*}GS(T-t)Ax]$$

$$= AS(s-t)x-L_t^s(I+L_t^*FL_t+L_t^{T*}GL_t^T)^{-1}[L_t^*FAS(\cdot-t)x+L_t^{T*}GAS(T-t)x].$$

Since FA and GA can be extended to be bounded on $L^2(\Omega)$ in view of (A.6), $L_t^*FAS(\cdot-t)x+L_t^{T*}GAS(T-t)x \in L^2([t,T]\times\Gamma)$ for $x \in L^2(\Omega)$. Moreover, we have

$$(5.8) \quad \left|(I+L_t^*FL_t+L_t^{T*}GL_t^T)^{-1}[L_t^*FAS(\cdot-t)x + L_t^{T*}GAS(T-t)x]\right|_{L^2([t,T]\times\Gamma)}$$

$$\leq c\,|x|_\Omega \text{ uniformly in } t \in [0,T] \text{ for } x \in L^2(\Omega).$$

On the other hand, since L_t is bounded from $L^2([t,T]\times\Gamma)$ into $C([t,T]; L^2(\Omega))$, we deduce from (5.8) that

(5.9) the second term in the right hand side of (5.7) is in $C([t,T]; L^2(\Omega))$ as a function of s, uniformly in $t \in [0,T]$.

It is clear that

$$S(\cdot - t)Ax \in C([t,T]; L^2(\Omega)) \text{ for } x \in D(A), \tag{5.10}$$

$$S(\cdot - t)Ax \in C([t,T]; [D(A^*)]') \text{ for } x \in L^2(\Omega),$$

uniformly in $t \in [0,T]$.

Hence, from (5.7), (5.9) and (5.10), we conclude that

$$\Phi(\cdot,t)Ax \in C([t,T]; L^2(\Omega)) \text{ for } x \in D(A), \tag{5.11}$$

$$\Phi(\cdot,t)Ax \in C([t,T]; [D(A^*)]') \text{ for } x \in L^2(\Omega)$$

uniformly in $t \in [0,T]$.

From (5.11) and lemma 5.1 (i), we arrive at

$$\Phi(\cdot,t)ADBP(t)x \in C([t,T]; [D(A^*)]') \tag{5.12}$$

for $x \in L^2(\Omega)$, uniformly in $t \in [0,T]$.

With the aid of (5.12), we now justify the following calculations in the space $[D(A^*)]'$ as in[1].

For $x \in L^2(\Omega)$ and a.e. $t \in [0,T]$, $0 \leq r < s \leq T$,

$$0 = \frac{d}{dt}\Phi(s,r)x = \frac{d}{dt}[\Phi(s,t)\Phi(t,r)x] \tag{5.13}$$

$$= [\frac{d}{dt}\Phi(s,t)]\Phi(t,r)x + \Phi(s,t)\frac{d}{dt}\Phi(t,r)x.$$

Since $\Phi(t,r)$ is the feedback evolution operator,

$$\frac{d}{dt}\Phi(t,r)x = A\Phi(t,r)x - ADBP(t)\Phi(t,r)x \quad \text{a.e. } t \in [r,T].$$

Thus, in view of (5.11) and (5.12), (5.13) becomes

$$(5.14) \quad 0 = [\frac{d}{dt}\Phi(s,t)]\Phi(t,r)x$$

$$+ \Phi(s,t)[A\Phi(t,r)x - ADBP(t)\Phi(t,r)x] \text{ in } [D(A^*)]'.$$

Hence, we have

$$(5.15) \quad [\frac{d}{dt}\Phi(s,t)]\Phi(t,r)x = -\Phi(s,t)A[I-DBP(t)]\Phi(t,r)x$$

for $x \in L^2(\Omega)$ and a.e. $t \in [0,T]$, $r \leqslant t \leqslant s$, where by (5.12), the right hand side is well defined in $[D(A^*)]'$.

Thus, the lemma follows from (5.15) after taking $r = t$.

Lemma 5.3. For $x \in L^2(\Omega)$ and $0 \leqslant t \leqslant s \leqslant T$,

$$\Phi(s,t)x = S(s-t)x - \int_t^s \Phi(s,r)ADBP(r)S(r-t)x dr.$$

Proof. From lemma 5.2, we have

$$(5.16) \quad [\frac{d}{dt}\Phi(s,t)]S(t-r)x \in [D(A^*)]' \text{ for } x \in L^2(\Omega)$$

and a.e. $t \in [0,T]$, $0 \leqslant r \leqslant t \leqslant s \leqslant T$. That is,

$$(5.17) \quad -\Phi(s,t)A[I-DBP(t)]S(t-r)x \in [D(A^*)]'$$

for $x \in L^2(\Omega)$ and $r \leqslant t \leqslant s$ uniformly in $t \in [0,T]$.

Since (5.17) is uniformly bounded in $t \in [0,T]$ and $[D(A^*)]'$ is complete, we obtain that

(5.18) $\int_{t_1}^{2} \Phi(s,t)A[I-DBP(t)]S(t-r)x\,dt$ is well defined in $[D(A^*)]'$ for $x \in L^2(\Omega)$ and $0 < r < t_1 < t_2 < s < T$.

That is, for $x \in L^2(\Omega)$ and $0 < r < t_1 < t_2 < s < T$,

$$\int_{t_1}^{t_2} [\frac{d}{dt} \Phi(s,t)]S(t-r)x\,dt \in [D(A^*)]'.$$

On the other hand, after integrating by parts, we obtain

$$(5.19) \quad \int_{t_1}^{t_2} [\frac{d}{dt} \Phi(s,t)]S(t-r)x\,dt$$

$$= [\Phi(s,t)S(t-r)]_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2} \Phi(s,t)AS(t-r)x\,dt$$

where integrations are performed in the space $[D(A^*)]'$. From (5.18), (5.19) becomes

$$(5.20) \quad \Phi(S,t_1)S(t_1-r)x = \Phi(s,t_2)S(t_2-r)x$$

$$- \int_{t_1}^{t_2} \Phi(s,t)ADBP(t)S(t-r)x\,dt$$

for $x \in L^2(\Omega)$ and $0 < r < t_1 < t_2 < s < T$.
By taking $t_1 = r$ and $t_2 = s$, we obtain, for $x \in L^2(\Omega)$,

$$(5.21) \quad \Phi(s,r)x = S(s-r)x - \int_r^s \Phi(s,t)ADBP(t)S(t-r)x\,dt.$$

Since $\Phi(s,r)x$ and $S(s-r)x$ are in $L^2(\Omega)$, the second term in the right hand side of (5.21) must be in $L^2(\Omega)$.

This completes the proof of the lemma.

Now we are in a position to prove theorem 5.1.

Proof of theorem 5.1(i). For x, $y \in L^2(\Omega)$ and $0 \leq t \leq T$, from (4.15) we have

$$(5.22) \quad (P(t)x,y)_\Omega = \int_t^T (\Phi(s,t)x,FS(s-t)y)_\Omega ds$$

$$+ (\Phi(T,t)x,GS(T-t)y)_\Omega.$$

By lemma 5.3, (5.22) becomes

$$(5.23) \quad (P(t)x,y)_\Omega = \int_t^T (S(s-t)x,FS(s-t)y)ds$$

$$+ (S(T-t)x,GS(T-t)y)_\Omega - I_1 - I_2$$

where $I_1 = \int_t^T (\int_t^s \Phi(s,r)ADBP(r)S(r-t)xdr,FS(s-t)y)_\Omega ds$

and $I_2 = (\int_t^T \Phi(T,r)ADBP(r)S(r-t)xdr,GS(T-t)y)_\Omega$.

In order to change the order of integrations in I_1 and I_2, we need to show that for x, $y \in L^2(\Omega)$,

$$(5.24) \quad \int_t^T \int_t^s (\Phi(s,r)ADBP(r)S(r-t)x,FS(s-t)y)_\Omega drds$$

and

$$(5.25) \quad \int_t^T (\Phi(T,r)ADBP(r)S(r-t)x,GS(T-t)y)_\Omega dr$$

are Lebesque integrals. From (5.12), for $x \in L^2(\Omega)$ we obtain

$$(5.26) \quad \Phi(s,r)ADBP(r)S(r-t)x \in [D(A^*)]'$$

uniformly bounded in r, s and t.

Thus, from (5.12) and (A.6), it follows that

$$(5.27) \quad (\Phi(s,r)ADBP(r)S(r-t)x,FS(s-t)y)_\Omega$$

and $$(\Phi(T,r)ADBP(r)S(r-t)x,GS(T-t)y)_\Omega$$

are well-defined and uniformly bounded in r, s and t. This implies that (5.24) and (5.25) are Lebesque integrals. Now we are in a position to apply Fubini's theorem [6] to I_1 and I_2. Then we have

$$(5.28)\quad I_1 = \int_t^T \int_t^s (\Phi(s,r)ADBP(r)S(r-t)x, FS(s-t)y)_\Omega drds$$

$$= \int_t^T \int_r^T (\Phi(s,r)ADBP(r)S(r-t)x, FS(s-r)S(r-t)y)_\Omega dsdr$$

$$= \int_t^T \int_r^T (S^*(s-r)F\Phi(s,r)ADBP(r)S(r-t)x, S(r-t)y)_\Omega dsdr,$$

$$(5.29)\quad I_2 = \int_t^T (\Phi(T,r)ADBP(r)S(r-t)x, GS(T-t)y)_\Omega dr$$

$$= \int_t^T (S^*(T-r)G\Phi(T,r)ADBP(r)S(r-t)x, S(r-t)y)_\Omega dr.$$

Similarly, from (5.12) and (A.6), it is easily seen that

$$\int_r^T S^*(s-r)F\Phi(s,r)ADBP(r)S(r-t)xds$$

is uniformly bounded in r.

Hence, by Fubini's theorem, from (5.28) we obtain

$$(5.30)\quad I_1 = \int_t^T \left(\int_r^T S^*(s-r)F\Phi(s,r)ADBP(r)S(r-t)xds,\right.$$

$$\left. S(r-t)y\right)_\Omega dr.$$

Combining (5.29) with (5.30), together with the remark in section 4.2, we have that

$$(5.31)\quad I_1+I_2 = \int_t^T (P(r)ADBP(r)S(r-t)x, S(r-t)y)_\Omega dr$$

$$= \int_t^T (BP(r)S(r-t)x, BP(r)S(r-t)y)_\Gamma dr$$

which is well defined in view of lemma 5.1 (i). Thus, the first part of theorem 5.1 follows from (5.23) and (5.31).

Proof of theorem 5.1 (ii). The Riccati differential equation is obtained by differentiating the Riccati integral equation with respect to t for x, y in D(A). In fact, for x, y in D(A) and a.e. t in [0,T], by differentiating the second Riccati integral equation with repspect to t, we obtain

$$
\begin{aligned}
(5.32)\quad \frac{d}{dt}\,(P(t)x,y)_{\Omega} &= -(x,Fy)_{\Omega}-\int_t^T (S(s-t)Ax,FS(s-t)y)_{\Omega}ds \\
&- \int_t^T (S(s-t)x,FS(s-t)Ay)_{\Omega}ds+(BP(t)x,BP(t)y)_{\Gamma} \\
&+ \int_t^T (BP(s)S(s-t)Ax,BP(s)S(s-t)y)_{\Gamma}ds \\
&+ \int_t^T (BP(s)S(s-t)x,BP(s)S(s-t)Ay)_{\Gamma}ds \\
&- (S(T-t)Ax,GS(T-t)y)_{\Omega} - (S(T-t)x,FS(T-t)Ay)_{\Omega}.
\end{aligned}
$$

All terms appearing in (5.32) are well defined in L^2-sense, in view of lemma 5.1 (i).

After applying the second Riccati integral equation to the inner products $(P(t)Ax,y)_{\Omega}$ and $(P(t)x,Ay)_{\Omega}$ in (5.32), it is easily seen that for x, y $\in$ D(A) and a.e. t $\in$ [0,T],

$$
\begin{aligned}
\frac{d}{dt}\,(P(t)x,y)_{\Omega} &= -(x,Fy)_{\Omega} - (P(t)Ax,y)_{\Omega} - (P(t)x,Ay)_{\Omega} \\
&+ (BP(t)x,BP(t)y)_{\Gamma} \text{ with } P(T) = G.
\end{aligned}
$$

This completes the proof of theorem 5.1.

5.2 Uniqueness of Riccati Operator

In this section, we give the proof of theorem 5.2. Let $\mathfrak{P}$ be the class of self-adjoint, positive-definite and bounded linear operators $\overline{P}(t)$, $0 < t < T$ on $L^2(\Omega)$ having the properties that

(5.33) $B\overline{P}(\cdot)$ is bounded from $L^2(\Omega)$ into $L^\infty([0,T]; L^2(\Gamma))$

and (5.34) $B\overline{P}(\cdot)S(\cdot)B^*$ is bounded from $L^2(\Gamma)$ into $L^2(\Sigma)$.

We show that the Riccati equations in theorem 5.1 admit the unique solution in $\mathfrak{P}$. Since the second Riccati integral equation can be derived from the Riccati differential equation in a usual way, the uniqueness for the Riccati differential equation follows from the uniqueness for the Riccati integral equation.
Thus, it is enough to prove the uniqueness for the second Riccati integral equation in the class $\mathfrak{P}$ (theorem 5.2).

Proof of theorem 5.2. Since our Riccati operator $P(t)$ defined in (4.15) is a solution to the Riccati equations in theorem 5.1 and it satisfies (5.33) and (5.34), the existence of a solution in $\mathfrak{P}$ is guaranteed.

Suppose that an operator $\overline{P}(t) \in \mathfrak{P}$ is also a solution to the second Riccati integral equation.

In fact, for $x, y \in L^2(\Omega)$ and $0 \leqslant t \leqslant T$,

$$(5.35) \quad (\overline{P}(t)x,y)_\Omega = \int_t^T (S(s-t)x, FS(s-t)y)_\Omega ds$$

$$- \int_t^T (B\overline{P}(s)S(s-t)x, B\overline{P}(s)S(s-t)y)_\Gamma ds + (S(T-t)x, GS(T-t)y)_\Omega .$$

We will show that $\overline{P}(t) = P(t)$ for $0 \leqslant t \leqslant T$. Let us consider the following mixed problem: for $y_o \in L^2(\Omega)$,

$$(5.36) \quad \begin{aligned} \partial_t y &= A(x,\partial)y && \text{in } Q \\ My &= B\overline{P}(\cdot)y(\cdot) && \text{in } \Sigma \\ y(0) &= y_o && \text{in } \Omega \end{aligned}$$

We first show that (5.36) has a unique solution in $L^2(Q)$. This can be done by Rauch's estimate (1.9) and a fixed point theorem. We rewrite (5.36), using semigroup representation, as

$$(5.37) \quad y = S(\cdot)y_o - LB\overline{P}(\cdot)y(\cdot).$$

Let us define an operator E as follows:

$$(5.38) \quad (Ey)(t) = (LB\overline{P}(\cdot)y(\cdot))(t), \ 0 \leqslant t \leqslant T,$$

$$\text{for } y \in D(E) \subset L^2(Q).$$

Then it is easily seen, from theorem 2.2 (i) and (5.33), that E is a bounded linear operator on $L^2(Q)$. By (5.38), (5.37) becomes

$$(5.39) \quad y = S(\cdot)y_o - Ey.$$

In order to show that (5.39) has a unique solution in $L^\infty([0,T]; L^2(\Omega))$, we show that E is a contraction map on $L^\infty([0,T]; L^2(\Omega))$ for small $T > 0$. For $y \in L^\infty([0,T]; L^2(\Omega))$, we have, from (5.33), that $B\overline{P}(\cdot)y(\cdot) \in L^\infty([0,T]; L^2(\Omega))$. Thus, by theorem 2.2 (i), it follows that

$$(5.40) \quad |Ey|_{L^\infty([0,T];\ L^2(\Omega))} \leq c\ |B\overline{P}(\cdot)y(\cdot)|_{L^2(\Sigma)}$$

$$\leq c'\ \sqrt{T}\ |B\overline{P}(\cdot)y(\cdot)|_{L^\infty([0,T];\ L^2(\Gamma))}$$

for $y \in L^\infty([0,T]; L^2(\Omega))$ where c and c' are constants independent on sufficiently small time T.

By (5.33), (5.40) becomes

$$|Ey|_{L^\infty([0,T];\ L^2(\Omega))} \leq c\ \sqrt{T}\ |y|_{L^\infty([0,T];\ L^2(\Omega)}$$

for $y \in L^\infty([0,T]; L^2(\Omega))$ where c is a constant independent on small T. Thus, E can be made a contraction map by taking $T_o > 0$ sufficiently small, i.e., $c\ \sqrt{T_o} < 1$.

By a fixed point theorem, (5.37) has a unique soultion $\overline{y}$ in $L^\infty([0,T_o]; L^2(\Omega))$ for small $T_o > 0$.

Since our system (5.36) is time-invariant and C^1, it can be shown that for any given $T > 0$, (5.36) has a unique solution by repeating the above argument n times where n is an integer such that $n\ T_o > T$. Let $\overline{u} = B\overline{P}(\cdot)\overline{y}(\cdot)$.

Our next step is to show that $\overline{u} = u^o$.

To this end, we introduce the evolution operator $\overline{\Phi}$ corresponding to (5.36) as in Chapter IV.

Then, as in Chapter IV, from (5.37), we have

$$(5.41)\quad \overline{\Phi}(s,t)x = S(s-t)x-(L_t B\overline{P}(\cdot)\overline{\Phi}(\cdot,t)x)(s)$$

for $x \in L^2(\Omega)$ and $0 \leq t \leq s \leq T$.
That is, for $x \in L^2(\Omega)$ and $0 \leq t \leq s \leq T$,

$$\overline{\Phi}(s,t)x = S(s-t)x-A\int_t^s S(s-r)DB\overline{P}(r)\overline{\Phi}(r,t)xdr.$$

Substituting (5.41) into (5.35), we obtain for $x, y \in L^2(\Omega)$,

$$\begin{aligned}(5.42)\quad (\overline{P}(t)x,y)_\Omega &= \int_t^T (\overline{\Phi}(s,t)x,FS(s-t)y)_\Omega ds\\
&+ \int_t^T (A\int_t^s S(s-r)DB\overline{P}(r)\overline{\Phi}(r,t)xdr,FS(s-t)y)_\Omega ds\\
&- \int_t^T (B\overline{P}(s)\overline{\Phi}(s,t)x,B\overline{P}(s)S(s-t)y)_\Gamma ds\\
&- \int_t^T (B\overline{P}(s)A\int_t^s S(s-r)DB\overline{P}(r)\overline{\Phi}(r,t)xdr,\\
&B\overline{P}(s)S(s-t)y)_\Gamma ds + (\overline{\Phi}(T,t)x,GS(T-t)y)_\Omega\\
&+ (A\int_t^T S(T-r)DB\overline{P}(r)\overline{\Phi}(r,t)xdr,GS(T-t)y)_\Omega\\
&= \int_t^T (S^*(s-t)F\overline{\Phi}(s,t)x,y)_\Omega ds\\
&+ (S^*(T-t)G\overline{\Phi}(T,t)x,y)_\Omega + I_1-I_2-I_3+I_4\end{aligned}$$

where $I_1 = \int_t^T (A\int_t^s S(s-r)DB\overline{P}(r)\overline{\Phi}(r,t)xdr,FS(s-t)y)_\Omega ds$,

$I_2 = \int_t^T (B\overline{P}(s)\overline{\Phi}(s,t)x,B\overline{P}(s)S(s-t)y)_\Gamma ds$,

$I_3 = \int_t^T (B\overline{P}(s)A\int_t^s S(s-r)DB\overline{P}(r)\overline{\Phi}(r,t)xdr,B\overline{P}(s)S(s-t)y)_\Gamma ds$

and $I_4 = (A\int_t^T S(T-r)DB\overline{P}(r)\overline{\Phi}(r,t)xdr,GS(T-t)y)_\Omega$.

Changing the order of integrations (this is justified in view of (A.6)), we obtain

$$(5.43) \quad I_1 = \int_t^T \int_t^s (S(s-r)B^*B\overline{P}(r)\overline{\Phi}(r,t)x, FS(s-t)y)_\Omega drds$$

$$= \int_t^T \int_r^T (S(s-r)B^*B\overline{P}(r)\overline{\Phi}(r,t)x, FS(s-t)y)_\Omega dsdr$$

and $(5.44) \quad I_4 = \int_t^T (S(T-r)B^*B\overline{P}(r)\overline{\Phi}(r-t)x, GS(T-t)y)_\Omega dr.$

Since $(B\overline{P}(s)\overline{\Phi}(s,t)x, B\overline{P}(s)S(s-t)y)_\Gamma = (\overline{P}(s)B^*B\overline{P}(s)\overline{\Phi}(s,t)x, S(s-t)y)_\Omega$, we obtain

$$(5.45) \quad I_2 = \int_t^T (\overline{P}(s)B^*B\overline{P}(s)\overline{\Phi}(s,t)x, S(s-t)y)_\Omega ds.$$

On the other hand, in order to use Fubini's theorem in I_3, we need to show that

$$\int_t^T \int_r^T (B\overline{P}(s)S(s-r)B^*B\overline{P}(r)\overline{\Phi}(r,t)x, B\overline{P}(s)S(s-t)y)_\Gamma dsdr$$

is Lebesque integral. This can be done as follows. In fact, by (5.33) and (5.34) together with Schwarz's inequality, we obtain the following inequalities;

$$\int_t^T \int_r^T \left|(B\overline{P}(s)S(s-r)B^*B\overline{P}(r)\overline{\Phi}(r,t)x, B\overline{P}(s)S(s-t)y)_\Gamma\right| dsdr$$

$$\leq \int_t^T \left[\int_r^T \left|B\overline{P}(s)S(s-r)B^*B\overline{P}(r)\overline{\Phi}(r,t)x\right|_\Gamma^2 ds\right]^{\frac{1}{2}}$$

$$\left[\int_r^T \left|B\overline{P}(s)S(s-t)y\right|_\Gamma^2 ds\right]^{\frac{1}{2}} dr$$

$$\leq \int_t^T c \left|B\overline{P}(r)\overline{\Phi}(r,t)x\right|_\Gamma |y|_\Omega dr \text{ for some constant } c > 0$$

$$\leq c' \, |x|_\Omega |y|_\Omega \text{ for some constant } c' > 0.$$

Thus, by Fubini's theorem, we have

$$(5.46) \quad I_3 = \int_t^T \int_t^s (B\overline{P}(s)S(s-r)B^*B\overline{P}(r)\overline{\Phi}(r,t)x, B\overline{P}(s)S(s-t)y)_\Gamma drds$$

$$= \int_t^T \int_r^T (B\overline{P}(s)S(s-r)B^*B\overline{P}(r)\overline{\Phi}(r,t)x, B\overline{P}(s)S(s-t)y)_\Gamma dsdr.$$

Let $u(s) = B^*B\overline{P}(s)\overline{\Phi}(s,t)x$ and $v(s) = S(s-t)y$.
Then $u(\cdot) \in L^\infty([t,T]; [D(A^*)]')$ and $v(\cdot) \in C([t,T]; L^2(\Omega))$.
We note that all integrals in (5.43) ~ (5.46) are Lebesque integrals. That is,

$$I_1 = \int_t^T \int_r^T (S(s-r)u(r), FS(s-r)v(r))_\Omega dsdr,$$

$$I_2 = \int_t^T (\overline{P}(r)u(r), v(r))_\Omega dr,$$

$$I_3 = \int_t^T \int_r^T (B\overline{P}(s)S(s-r)u(r), B\overline{P}(s)S(s-r)v(r))_\Gamma dsdr$$

and $I_4 = \int_t^T (S(T-r)u(r), GS(T-r)v(r))_\Omega dr$

are all Lebesque integrals.

Combining I_1, I_2, I_3 and I_4 , we obtain

$$(5.47) \quad I_1-I_2-I_3+I_4 = \int_t^T [\int_r^T (S(s-r)u(r), FS(s-r)v(r))_\Omega ds - (\overline{P}(r)u(r), v(r))_\Omega - \int_r^T (B\overline{P}(s)S(s-r)u(r), B\overline{P}(s)S(s-r)v(r))_\Gamma ds + (S(T-r)u(r), GS(T-r)v(r))_\Omega]dr.$$

On the other hand, in view of (5.33) and (5.34), the second Riccati integral equation holds for $x \in [D(A^*)]'$ and $y \in L^2(\Omega)$, thus, in particular for $u(r)$ and $v(r)$, a.e. $r \in [t,T]$.

Applying the second Riccati integral equation to (5.47) with $x = u(r)$ and $y = v(r)$ a.e. $r \in [t,T]$, we obtain

$$(5.48) \quad I_1 - I_2 - I_3 + I_4 = 0.$$

From (5.42) and (5.48), it is readily seen that

$$(5.49) \quad (\overline{P}(t)x,y)_\Omega = \int_t^T (S^*(s-t)F\overline{\Phi}(s,t)x,y)_\Omega ds$$
$$+ (S^*(T-t)G\overline{\Phi}(T,t)x,y)_\Omega \text{ for all } x,\ y \text{ in } L^2(\Omega)$$

and t in $[0,T]$. Thus, from (5.49), we obtain

$$(5.50) \quad \overline{P}(t)x = \int_t^T S^*(s-t)F\overline{\Phi}(s,t)x ds + S^*(T-t)G\overline{\Phi}(T,t)x$$
$$\text{for all } x \text{ in } L^2(\Omega) \text{ and } t \text{ in } [0,T].$$

Since $\overline{\Phi}$ is the evolution operator for (5.36), for $t < s$, $\overline{y}(s) = \overline{\Phi}(s,t)\overline{y}(t)$. Therefore, (5.50) becomes

$$(5.51) \quad \overline{P}(t)\overline{y}(t) = \int_t^T S^*(s-t)F\overline{\Phi}(s,t)\overline{y}(t)ds$$
$$+ S^*(T-t)G\overline{\Phi}(T,t)\overline{y}(t)$$
$$= \int_t^T S^*(s-t)F\overline{y}(s)ds + S^*(T-t)G\overline{y}(T).$$

From (5.51), we obtain

$$(5.52) \quad \overline{u}(t) = B\overline{P}(t)\overline{y}(t) = B\int_t^T S^*(s-t)F\overline{\Phi}(s,t)\overline{y}(t)ds$$
$$+ BS^*(T-t)G\overline{\Phi}(T,t)\overline{y}(t)$$
$$= (L^*F\overline{y})(t) + (L^{T*}G\overline{y}(T))(t) \text{ for a.e. } t \in [0,T].$$

From (3.7), (5.52) implies that $\overline{u}$ satisfies the optimality condition for our control problem.

By the uniqueness of the optimal control to our control problem, we conclude that $\overline{u} = u^o$.
Thus, $\overline{y} = y^o$ and consequently, $\overline{\Phi} = \Phi$. By (4.15) and (5.50), we conclude that $\overline{P}(t) = P(t)$ for all $t \in [0,T]$.

This completes the proof of theorem 5.2.

Remark. We use (5.33) and (5.34) to justify change of the order of integrations in I_1, I_2, I_3 and I_4, and also to verify the second Riccati integral equations for $u(r)$ and $v(r)$ a.e. r.

5.3 Regularization

In this section, we assume that F and G are arbitrary self-adjoint, positive-definite bounded operators on $L^2(\Omega)$; i.e., we do not assume any smoothness on F and G.

We will show that the corresponding Riccati operator defined in (4.15) can be determined as the strong limit on $L^2(\Omega)$ of a sequence of solutions to regularized Riccati equations in theorem 5.1. In fact, the purpose of this section is to give the proof of theorem 5.3.

Suppose that $\{F_n\}_n$ and $\{G_n\}_n$ are sequences of self-adjoint, positive-definite operators on $L^2(\Omega)$ which satisfy

(5.53) F_n and G_n map $L^2(\Omega)$ into $D(A^*)$ for all n,

$F_n \to F$ and $G_n \to G$ strongly in $L^2(\Omega)$ as $n \to \infty$.

Such sequences can be easily constructed as we will see later. For simplicity of notation, let us denote by u_n^o, y_n^o, Φ_n, P_n, etc., the optimal control, trajectory, evolution operator, Riccati operator, etc., corresponding to the smoothing operators F_n and G_n in the functional cost (1.12).

In order to prove theorem 5.3, we need the following lemma.

Lemma 5.4. For any given $x \in L^2(\Omega)$,

(i) $u_n^o(\cdot,t;x) \to u^o(\cdot,t;x)$ in $L^2([t,T]x\Gamma)$ as $n \to \infty$ uniformly in t, $0 \leq t \leq T$,

(ii) $\Phi_n(s,t)x \to \Phi(s,t)x$ in $L^2(\Omega)$ as $n \to \infty$ uniformly in both s and t, $0 \leq t \leq s \leq T$.

Proof of part (i). From lemma 4.2, we recall that for each n and $0 \leq t \leq T$,

$$(5.54) \quad u_n^o(\cdot,t;x) = [I+L_t^*F_nL_t+L_t^{T*}G_nL_t^T]^{-1}[L_t^*F_nS(\cdot-t)x + L_t^{T*}G_nS(T-t)x].$$

Let $R_n = I+L_t^*F_nL_t+L_t^{T*}G_nL_t^T$ and $R_o = I+L_t^*FL_t+L_t^{T*}GL_t^T$. We will show that $R_n^{-1} \to R_o^{-1}$ strongly in $L^2([t,T]x\Gamma)$ as $n \to \infty$ uniformly in $t \in [0,T]$.

For a given $u \in L^2([t,T]x\Gamma)$, we have

$$(5.55) \quad |R_nu-R_ou|_{[t,T]x\Gamma} \leq |L_t^*(F_nL_tu-FL_tu)|_{[t,T]x\Gamma} + |L_t^{T*}(G_nL_t^Tu-GL_t^Tu)|_{[t,T]x\Gamma}.$$

In view of lemma 4.1 (i), (ii), it is easily seen that L_t^* and L_t^{T*} are uniformly bounded in t. Thus, (5.55) becomes

$$(5.56) \quad |R_n u - R_o u|_{[t,T]\times\Gamma} < c\, |F_n L_t u - F L_t u|_{[t,T]\times\Omega}$$

$$+ c\, |G_n L_t^T u - G L_t^T u|_\Omega \text{ for some constant } c > 0.$$

We recall that strong convergences of bounded linear operators on a Hilbert space turn to be uniform convergences on compact subsets of the space. Thus, from (5.53), $F_n \to F$ and $G_n \to G$ uniformly on compact subsets of $L^2(\Omega)$ as $n \to \infty$. On the other hand, from lemma 4.1 (ii), we deduce that $\{L_t^T u \mid 0 < t < T\}$ and $\{L_t^s u \mid 0 < t < s < T\}$ are compact subsets of $L^2(\Omega)$, being the images of compact sets under continuous maps. Hence, we obtain

$$(5.57) \quad |G_n L_t^T u - G L_t^T u|_\Omega \to 0 \text{ as } n \to \infty \text{ uniformly in } t \in [0,T]$$

$$\text{and} \quad |F_n L_t^s u - F L_t^s u|_\Omega \to 0 \text{ as } n \to \infty \text{ uniformly in}$$

$$\text{both } s \text{ and } t,\ 0 < t < s < T.$$

Combining (5.56) with (5.57), we conclude

$$(5.58) \quad |R_n u - R_o u|_{[t,T]\times\Gamma} \to 0 \text{ as } n \to \infty \text{ uniformly in } t;$$

i.e., $R_n \to R_o$ strongly in $L^2([t,T]\times\Gamma)$ as $n \to \infty$ uniformly in t. For $u \in L^2([t,T]\times\Gamma)$,

$$|R_n^{-1} u - R_o^{-1} u|_{[t,t]\times\Gamma} = |R_n^{-1}(R_n - R_o) R_o^{-1} u|_{[t,T]\times\Gamma}$$

$$< |(R_n - R_o) R_o^{-1} u|_{[t,T]\times\Gamma} \to 0 \text{ as } n \to \infty \text{ uniformly in } t,$$

where in the last inequality, we use the fact that $\|R_n^{-1}\| \leq 1$ for all n, and (5.58). Thus, we obtain

$$(5.59) \quad R_n^{-1} \to R_o^{-1} \text{ strongly in } L^2([t,T]\times\Gamma) \text{ as } n \to \infty$$
$$\text{uniformly in } t \in [0,T].$$

Let
$$v_n = L_t^* F_n S(\cdot - t)x + L_t^{T*} G_n S(T-t)x,$$
$$v = L_t^* F S(\cdot - t)x + L_t^{T*} G S(T-t)x.$$

Then in a similar manner, it is easily seen that

$$(5.60) \quad v_n \to v \text{ in } L^2([t,T]\times\Gamma) \text{ as } n \to \infty \text{ uniformly in } t.$$

From (5.54), we obtain

$$(5.61) \quad |u_n^o(\cdot,t;x) - u^o(\cdot,t;x)|_{[t,T]\times\Gamma} = |R_n^{-1}v_n - R_o^{-1}v|_{[t,T]\times\Gamma}$$
$$\leq |R_n^{-1}v_n - R_n^{-1}v|_{[t,T]\times\Gamma} + |R_n^{-1}v - R_o^{-1}v|_{[t,T]\times\Gamma}$$
$$\leq \|R_n^{-1}\|\,|v_n - v|_{[t,T]\times\Gamma} + |R_n^{-1}v - R_o^{-1}v|_{[t,T]\times\Gamma}.$$

Since $\|R_n^{-1}\| \leq 1$ for all n, by (5.59) and (5.60), the right hand side of (5.61) converges to 0 as $n \to \infty$ uniformly in $t \in [0,T]$.

This completes the proof of part (i).

Proof of Part (ii). For $t \leq s$ and $x \in L^2(\Omega)$, from (4.7) we obtain

$$(5.62)\quad \Phi_n(s,t)x-\Phi(s,t)x = (L_t(u_n^o(\cdot,t;x)-u^o(\cdot,t;x)))(s).$$

Since L_t is bounded from $L^2([t,T]x\Gamma)$ into $C([t,T];\ L^2(\Omega))$ uniformly in t, (5.62) yields

$$(5.63)\quad |\Phi_n(s,t)x-\Phi(s,t)x|_\Omega$$
$$\leq |L_t(u_n^o(\cdot,t;x)-u^o(\cdot,t;x))|_{C([t,T];\ L^2(\Omega))}$$
$$\leq c\ |u_n^o(\cdot,t;x)-u^o(\cdot,t;x)|_{L^2([t,T]x\Gamma)}$$

for some constant $c > 0$ where the last term converges to 0 as $n \to \infty$ uniformly in t by part (i).

This completes the proof of part (ii).

Remark. From lemma 5.4, we infer that

(i) $u_n^o \to u^o$ in $L^2(\Sigma)$ as $n \to \infty$,

(ii) $y_n^o(t) \to y^o(t)$ uniformly in t as $n \to \infty$.

Now we are in a position to prove theorem 5.3.

Proof of theorem 5.3. For any $t \in [0,T]$ and $x \in L^2(\Omega)$, we have

$$(5.64)\quad |P_n(t)x-P(t)x|_\Omega$$
$$\leq \int_t^T |S^*(s-t)[F_n\Phi_n(s,t)x-F\Phi(s,t)x]|_\Omega ds$$
$$+ |S^*(T-t)[G_n\Phi_n(T,t)x-G\Phi(T,t)x]|_\Omega.$$

Since $S^*(t)$ is a strongly continuous semigroup, its norm is bounded on a finite time interval. Thus, (5.64) becomes

(5.65) $|P_n(t)x-P(t)x|_\Omega \leq c \int_t^T |F_n\Phi_n(s,t)x-F\Phi(s,t)x|_\Omega ds$

$+ c\ |G_n\Phi_n(T,t)x-G\Phi(T,t)x|_\Omega \leq c \int_t^T |F_n\Phi_n(s,t)x-F_n\Phi(s,t)x|_\Omega ds$

$+ c \int_t^T |F_n\Phi(s,t)x-F\Phi(s,t)x|_\Omega ds + c\ |G_n\Phi_n(T,t)x-G_n\Phi(T,t)x|_\Omega$

$+ c\ |G_n\Phi(T,t)x-G\Phi(T,t)x|_\Omega$

for some constant $c > 0$. From lemma 4.3 (iii) and (v), we deduce that

(5.66) $\{\Phi(s,t)x \mid 0 \leq t \leq s \leq T\}$ and

$\{\Phi(T,t)x \mid 0 \leq t \leq T\}$ are compact subsets of $L^2(\Omega)$.

Thus, by the similar argument to the proof of (i) in lemma 5.4, we obtain that by (5.66),

(5.67) $|F_n\Phi(s,t)x-F\Phi(s,t)x|_\Omega \to 0$

uniformly in both s and t as $n \to \infty$,

and $|G_n\Phi(T,t)x-G\Phi(T,t)x|_\Omega \to 0$

uniformly in t as $n \to \infty$.

From lemma 5.4 (ii) and (5.67), it is clear that the right hand side of (5.65) converges to 0 as $n \to \infty$ uniformly in t. Thus, from (5.65), we conclude that $|P_n(t)x-P(t)x|_\Omega \to 0$ as $n \to \infty$ uniformly in $t \in [0,T]$. That is, $P_n(t) \to P(t)$ strongly in $L^2(\Omega)$ as $n \to \infty$ uniformly in t. From the Riccati integral equation in theorem 5.1, it can be easily seen that for $x, y \in L^2(\Omega)$,

$$(P(t)x,y)_\Omega = \int_t^T (S(s-t)x,FS(s-t)y)_\Omega ds$$
$$- \lim_{n\to\infty} \int_t^T (BP_n(s)S(s-t)x,BP_n(s)S(s-t)y)_\Gamma ds$$
$$+ (S(T-t)x,GS(T-t)y)_\Omega \quad \text{uniformly in } t.$$

This completes the proof of theorem 5.3.

Finally, we are in a position to complete the proof of theorem 4.1 without (A.6).

Proof of theorem 4.1 (ii). From theorem 4.1 (ii) under (A.6), we obtain that for $x, y \in L^2(\Omega)$, $t \in [0,T]$,

$$(5.68) \quad (P_n(t)x,y)_\Omega = \int_t^T(\Phi_n(s,t)x,F_n\Phi_n(s,t)y)_\Omega ds$$
$$+ \int_t^T (BP_n(s)\Phi_n(s,t)x,BP_n(s)\Phi_n(s,t)y)_\Gamma ds$$
$$+ (\Phi_n(T,t)x,G_n\Phi_n(T,t)y)_\Omega.$$

From theorem 5.3, we have

$$(5.69) \quad (P_n(t)x,y)_\Omega \to (P(t)x,y)_\Omega \text{ as } n \to \infty \text{ uniformly in } t.$$

By lemma 5.4 (ii), it is clear that

$$(5.70) \quad \int_t^T (\Phi_n(s,t)x,F_n\Phi_n(s,t)y)_\Omega ds$$
$$\to \int_t^T (\Phi(s,t)x,F\Phi(s,t)y)_\Omega ds \text{ as } n \to \infty,$$

and $(\Phi_n(T,t)x,G_n\Phi_n(T,t)y)_\Omega \to (\Phi(T,t)x,G\Phi(T,t)y)_\Omega$ as $n \to \infty$. From (4.17), we recall that for $x \in L^2(\Omega)$, $t \in [0,T)$, $u^o(\cdot,t;x) = BP(\cdot)\Phi(\cdot,t)x$ and $u_n^o(\cdot,t;x) = BP_n(\cdot)\Phi_n(\cdot,t)x$. By lemma 5.4 (i), it is easily seen that

$$(5.71)\quad (u_n^o(\cdot,t;x),u_n^o(\cdot,t;y))_{[t,T]\times\Gamma}$$

$$\to (u^o(\cdot,t;x),u^o(\cdot,t;y))_{[t,T]\times\Gamma} \text{ as } n \to \infty.$$

In fact,

$$|(u_n^o(\cdot,t;x),u_n^o(\cdot,t;y))_{[t,T]\times\Gamma}-(u^o(\cdot,t;x),u^o(\cdot,t;y))_{[t,T]\times\Gamma}|$$

$$\leq |(u_n^o(\cdot,t;x),u_n^o(\cdot,t;y))_{[t,T]\times\Gamma}-(u_n^o(\cdot,t;x),u^o(\cdot,t;y))_{[t,T]\times\Gamma}|$$

$$+ |(u_n^o(\cdot,t;x),u^o(\cdot,t;y))_{[t,T]\times\Gamma}- (u^o(\cdot,t;x),u^o(\cdot,t;y))_{[t,T]\times\Gamma}|$$

$$= |(u_n^o(\cdot,t;x),u_n^o(\cdot,t;y)-u^o(\cdot,t;y))_{[t,T]\times\Gamma}|$$

$$+ |(u_n^o(\cdot,t;x)-u^o(\cdot,t;x),u^o(\cdot,t;y))_{[t,T]\times\Gamma}|$$

$\to 0$ as $n \to \infty$ which follows from lemma 5.4 (i).

Thus, we may rewrite (5.71) as

$$(5.72)\quad \int_t^T (BP_n(s)\Phi_n(s,t)x,BP_n(s)\Phi_n(s,t)y)_\Gamma ds$$

$$\to \int_t^T (BP(s)\Phi(s,t)x,BP(s)\Phi(s,t)y)_\Gamma ds \text{ as } n \to \infty.$$

Hence, by taking limits in (5.68), we obtain

$$(P(t)x,y)_\Omega = \int_t^T (\Phi(s,t)x,F\Phi(s,t)y)_\Omega ds$$

$$+ \int_t^T (BP(s)\Phi(s,t)x,BP(s)\Phi(s,t)y)_\Gamma ds + (\Phi(T,t)x,G\Phi(T,t)y)_\Omega.$$

This completes the proof of theorem 4.1 (ii).

From theorem 4.1 (ii), we obtain the following corollary without any smoothness assumptions on F and G.

Corollary. For each $t \in [0,T)$, we have

(i) $P(t)$ is self-adjoint and positive-definite,

(ii) the minimal value of the functional cost J_t on the interval $[t,T]$ with initial data x is given by $J_t(u^o(\cdot,t;x),y^o(\cdot,t;x)) = \frac{1}{2}(P(t)x,x)_\Omega$.

We give examples of how to obtain sequences of smoothing operators F_n and G_n converging to F and G strongly in $L^2(\Omega)$ respectively.

Example 1. Since the operators A and A^* generate strongly continuous semigroups on $L^2(\Omega)$, there exists a number ω_o such that $\{r \mid r > \omega_o\}$ is contained in the resolvent sets of A and A^* [1]. Thus, the operator $(n+\omega_o-A^*)^{-1}(n+\omega_o-A)^{-1}$ is bounded on $L^2(\Omega)$ for each positive integer n.

Let $R_n = (n+\omega_o)^2(n+\omega_o-A^*)^{-1}(n+\omega_o-A)^{-1}$ for each n. Then it is clear that R_n is self-adjoint and positive-definite for each n. Moreover, R_n maps $L^2(\Omega)$ into $D(A^*)$ for each n.

On the other hand, it is well known [1] that $R_n \to I$ strongly in $L^2(\Omega)$ as $n \to \infty$. Let $F_n = R_nF$ and $G_n = R_nG$ for each n. Then F_n and G_n are self-adjoint, positive-definite and bounded operators on $L^2(\Omega)$ which map $L^2(\Omega)$ into $D(A^*)$ for each n.

For each $x \in L^2(\Omega)$, we have

$$|F_n x - Fx|_\Omega = |R_n Fx - Fx|_\Omega \to 0 \text{ and}$$

$$|G_n x - Gx|_\Omega = |R_n Gx - Gx|_\Omega \to 0 \text{ as } n \to \infty.$$

This implies that $\{F_n\}_n$ and $\{G_n\}_n$ satisfy (5.53).

Example 2. Let $A_d = \Delta$ with zero boundary condition on Ω, where $\Delta = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}$ with domain $D(A_d) = H^2(\Omega) \cap H_o^1(\Omega)$. Then it is clear that $-A_d$ is self-adjoint and positive-definite. Thus, $-A_d^{\frac{1}{2}}$ is also self-adjoint and positive-definite with domain $D(-A_d^{\frac{1}{2}}) = H_o^1(\Omega)$.

It is well known that $A_d^{\frac{1}{2}}$ generates a strongly continuous semigroup on $L^2(\Omega)$. Thus, for some real number ω_o, the set $\{r \mid r > \omega_o\}$ is contained in the resolvent set of $A_d^{\frac{1}{2}}$. That is, $(n+\omega_o-A_d^{\frac{1}{2}})^{-1}$ is bounded on $L^2(\Omega)$ for each positive integer n. Let $R_n = (n+\omega_o)(n+\omega_o-A_d^{\frac{1}{2}})^{-1}$ for integer $n > 1$. Then, since $-A_d^{\frac{1}{2}}$ is self-adjoint and positive-definite, so is R_n for each $n > 0$.
Moreover, R_n maps $L^2(\Omega)$ into $H_o^1(\Omega)$ for each n.

It is well known [1] that $R_n \to I$ strongly in $L^2(\Omega)$ as $n \to \infty$. Let $F_n = R_n F$ and $G_n = R_n G$ for each n. As in example 1, it is easily seen that $\{F_n\}_n$ and $\{G_n\}_n$ satisfy (5.53).

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BIOGRAPHICAL SKETCH

Sung Kag Chang was born on June 6, 1951, at a quiet and traditional village in Korea. He is the third among four sons of Mr. Chun Sik and Mrs. Soo Sun Chang. Sung attended high school and college in Taegu, receiving his B.S. and M.S. from Kyungpook National University, in 1974 and 1976, respectively. He came to the University of Florida in 1979. While in Florida, he has received his M.S. in applied mathematics and also married Oe Im Chang in the summer of 1981. During the last five years he has learned and received so much, most important, his wonderful daughter, Jennifer.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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